Plugstars: Alternative to Black Holes.

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Abstract: We focus on the only directly usable data ftom images of hypermassive objects at the center of the M 87 and Milky Way galaxies, namely the ratios of maximum to minimum temperatures, in both cases very close to 3. To explain that their center remain emissive, it has been suggested that there is some hot gaz, belonging to an accretion disk, located at the foreground. But as this hypothetical accretion disk does not extend beyond the images, their identification as giant black holes is questionable. After examining what led to the emergence of the black hole theory, and evoking the alternative of gravstars, we show that the plugstars model, where the darkening of the central part is then attributed to a subcritical situation, is the only one that fits perfectly available observational data.

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1 – **Introduction.**

In 1979, 63 years after the publication of his two articles by Karl Schwarzschild, L.S.Abrams published an article entitled “The Legacy of Hilbert's error” [1]. This theme was subsequently taken up by S.Antoci and E.E. Liebscher in 2003 [2]. These authors denounce the alleged confusion between the original solution published by Karl Schwarzschild in 2016 [3]. In 2011, C.Corda [4] countered these criticisms by claiming that the so-called errors were based solely on a misinterpretation of the variables. It all hinges on the fact that the various authors, Droste and Hilbert first, relying on the fact that solutions to Einstein's equation can be formulated in any coordinate system, allow themselves one or more successive changes of a supposed radial coordinate under the pretext that the solution identifies with the Lorentz metric at infinity.

(1) $$ds^2 = c^2dt^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

What has been underestimated is the autonomy and intelligence of the solutions emanating from this equation, revealing an underlying topology. This was identified as early as 1917 by Herman Weyl, who wrote [6], page 794, we quote:

Dieses Linienelement charakterisiert die Geometrie, die auf dem folgenden Rotationsparaboloid im Euklidischen Raum mit den rechtwinkligen Koordinaten x_1, x_2, z gilt:

$$z = \sqrt{8a(r - 2a)},$$

wenn dasselbe durch orthogonale Projektion auf die Ebene $z = 0$ mit den Polarkoordinaten r, ϑ bezogen wird. Die Projektion bedeckt das Äußere des Kreises $r \geqq 2a$ doppelt, das Innere überhaupt nicht. Bei natürlicher analytischer Fortsetzung wird also der wirkliche Raum in dem zur Darstellung benutzten Koordinatenraum der x_i das durch $r \geqq 2a$ gekennzeichnete Gebiet doppelt überdecken. Die beiden Überdeckungen sind durch die Kugel $r = 2a$, auf der sich die Masse befindet und die Maßbestimmung singulär wird, geschieden, und man wird jene beiden Hälften als das „Äußere" und das „Innere" des Massenpunktes zu bezeichnen haben.

Fig.1 : Weyl, meridian equation.

Translation :

This line element characterizes the geometry that is valid on the paraboloid of rotation

$$z = \sqrt{8a(r - 2a)}$$

in a Euclidean space with the orthogonal coördinates x_1, x_2, z if the paraboloid is projected orthogonally onto the $z = 0$ plane with the polar coördinates r, ϑ. The projection covers the exterior of the circle $r \geq 2a$ twice, but does not cover the interior at all. Via natural analytic continuation, the true space will cover the domain $r \geq 2a$ doubly in the coördinate space of the x_i used to represent it. The two coverings are separated by the sphere $r = 2a$ on which the mass lies and at which the metric becomes singular and one has to refer to the two halves as the "outside" and "inside" of the point mass.

Fig.1bis : Weyl, meridian equation.

Unlike Schwarzschild, Weyl considers a structure that is entirely described by the solution of Einstein's zero second-member equation, with a view to giving masses a topological nature that reflects a connection with a second sheet of space-time. Here we see the concept of a two-sheet covering of an edge variety, the latter being the Schwarzschild sphere. Not surprisingly, the hypersurface described by this non-contractile metric has a minimal perimeter that is 2π multiplied by the Schwarschild length (here 2a).

This topological structure was also described in 1916 by the mathematician Ludwig Flamm [7], an article whose English translation only became available in 2012, a year after the publication of Corda's article [4]. Remarkably, this young mathematician had immediately given the correct interpretation of the set of two solutions that Karl Schwarzschild, a mathematician, geometer, physicist and astronomer, had just published ([3],[8]). Indeed, Schwarzschild's description of the geometry outside and inside a sphere filled with incompressible material of constant density corresponds to the connection of a sphere S3 with what we might call Flamm's 3D hypersurface, along a sphere S2. In this figure, Flamm traces the meridian associated with this 3D structure, consisting of a circular arc connecting with a portion of a parabola.

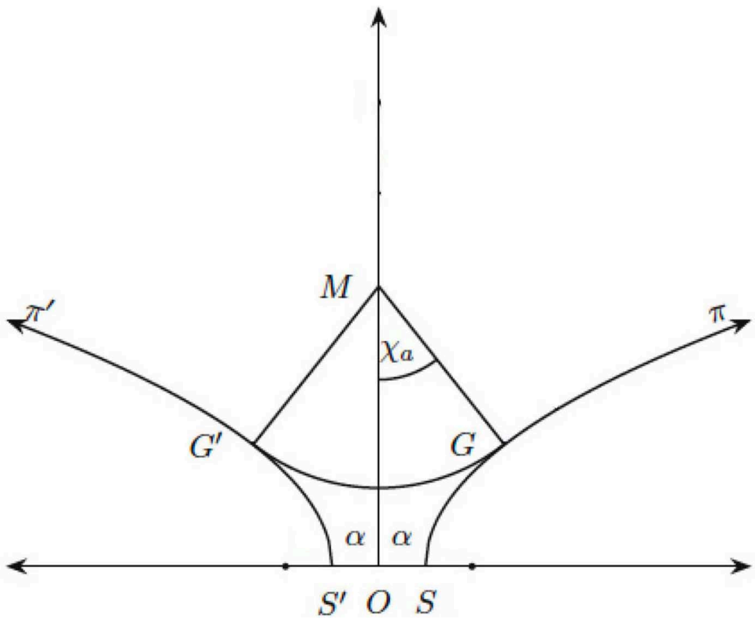

Fig.2 : The meridian of the 3D hypersurface solution.

The “Flamm surface” obtained by rotating this supine parabola around its axis provides a didactic 2D image of this non-contractile 3D hypersurface. Lets’ quote Flamm’s text [7]:

> *The mass point, which generates the gravitational field, is found as the vertex S of the meridional parabola. The surface of rotation of the Branch* $S\pi$ *of the parabola, as*

seen in the figure, already maps to the full sectional plane thgrough the centre, preserving the metric properties. The particularity that the point mass has a finite circumference of length $2\pi\alpha$*, as Schwarzschild has already emphasized, is clearly noticeable in the figure.*

End of quote.

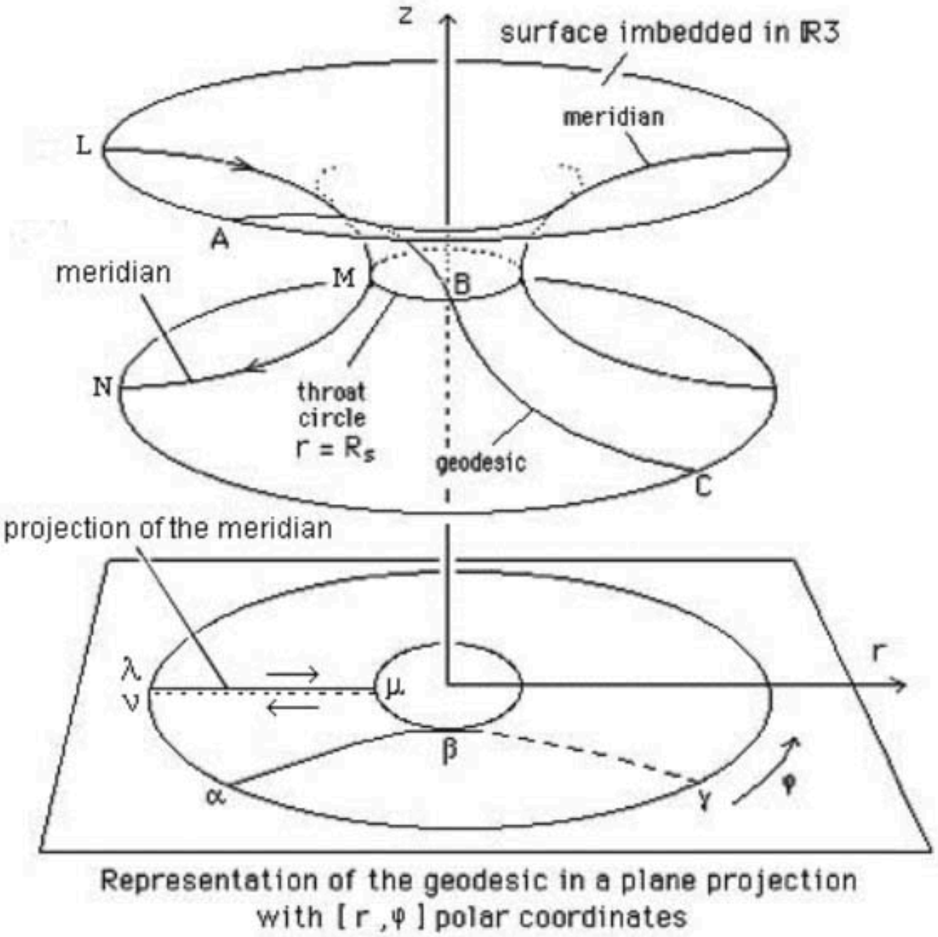

Fig.3 : Flamm surface [10]

But it would be wrong to confuse the geodesics of this 2-surface with the geodesics of the 3-surface connecting with the portion of sphere S3 along a sphere S2 . Flamm takes up the complete work of K. Schwarzschild, but does not rule out considering the non-contractile 3D hypersurface resulting from the external metric, considered in isolation, as the representation of a mass, as Weyl does, and as Einstein and Rosen would later do in 1935 [9]. Figure 3, taken from [10], also illustrates the topology derived from the Schwarzschild outer metric. This illustration is all the more telling in that it relies on a new variable which, on its own, can describe both layers, depending on the change of variable, applied to what is considered the "standard formulation of the Schwarzschild solution":

(2) $$ds^2 = \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{dR^2}{1 - \frac{\alpha}{R}} - R^2 d\theta^2 - R^2 \sin^2\theta d\varphi$$

This change of coordinate being [10]:

(3) $$R = \alpha \left[1 + L_n \operatorname{ch}\rho \right]$$

Which gives :

(4) $$ds^2 = \frac{L_n \operatorname{ch}\rho}{1 + L_n \operatorname{ch}\rho} c^2 dt^2 - R_s^2 \frac{1 + L_n \operatorname{ch}\rho}{L_n \operatorname{ch}\rho} \operatorname{th}^2\rho \, d\rho^2 - R_s^2 \left(1 + L_n \operatorname{ch}\rho \right)^2 \left(d\theta^2 + \sin^2\theta d\varphi^2 \right)$$

This formulation then describes the object in its entirety, integrating its non-contractibility. It is made up of two layers, one for ρ varying from $+\infty$ to 0 and the other from 0 to $-\infty$, connecting on the Schwarzschild sphere, corresponds to the value $\rho = 0$. The above elements show that C.Corda [4] did not understand the topological nature of the problem. Correctly interpreted, the object is non-contractile and free of central singularity.

2 – **Space-time according to Hilbert.**

Historians of science have published meticulous analyses [11] of these articles by David Hilbert ([12],[13]). It's worth putting them into context. At the start of his career, Hilbert could not imagine that the mathematics in which he evolved, the "pure mathematics" that seemed to him to be pure abstraction, could have such close links with physics. But his attitude subsequently changed, particularly in the course of exchanges with Einstein, whom he invited to give lectures in Göttingen in June and July 1915, and he began to apply the tools of modern geometry to physics. He thus succeeded in constructing a field equation, which he published on November 20, 1915 [12], four days before Einstein published his own. This is on page 404 of [12], in equation (21):

Unter Verwendung der vorhin eingeführten Bezeichnungsweise für die Variationsableitungen bezüglich der $g^{\mu\nu}$ erhalten die Gravitationsgleichungen wegen (20) die Gestalt

(21) $$[\sqrt{g}\,K]_{\mu\nu} + \frac{\partial \sqrt{g}\,L}{\partial g^{\mu\nu}} = 0.$$

Das erste Glied linker Hand wird

$$[\sqrt{g}\,K]_{\mu\nu} = \sqrt{g}\left(K_{\mu\nu} - \tfrac{1}{2} K g_{\mu\nu}\right),$$

Fig.4 : D.Hilbert's field equation [13].

Translation:

Using the notation introduced earlier for the variational derivatives with respect to the $g^{\mu\nu}$, the gravitational equations, because of (20), take the form

$$[\sqrt{g}K]_{\mu\nu} + \frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} = 0. \qquad (21)$$

The first term on the left hand side becomes

$$[\sqrt{g}K]_{\mu\nu} = \sqrt{g}\left(K_{\mu\nu} - \frac{1}{2}K g_{\mu\nu}\right),$$

Fig.4bis : D.Hilbert's field equation [13]

The tensor $K_{\mu\nu}$ is the Ricci tensor and the scalar K the Ricci scalar. By writing

(5) $$T_{\mu\nu} = -\frac{\partial\sqrt{g}\,L}{\partial g^{\mu\nu}}$$

we find the general relativity equation. We will, of course, focus on Hilbert's treatment of the spherically symmetric stationary solution of Einstein's equation, which refers to his second paper ([12],). As in his first article, Hilbert presents his own understanding of special relativity on the very first page of his paper:

Zunächst führen wir an Stelle der Weltparameter w_s (s = 1, 2, 3, 4) die allgemeinsten reellen Raum-Zeit-Koordinaten x_s (s = 1, 2, 3, 4) ein, indem wir

$$w_1 = x_1, \quad w_2 = x_2, \quad w_3 = x_3, \quad w_4 = i x_4$$

Fig.5 : Hilbert's space-time coordinates [13].

Translation :

First we introduce in place of the world parameters w_s ($s = 1, 2, 3, 4$) the most general *real* spacetime coordinates x_s ($s = 1, 2, 3, 4$) by putting

$$w_1 = x_1, \quad w_2 = x_2, \quad w_3 = x_3, \quad w_4 = \mathrm{i} x_4,$$

Fg.5bis : Hilbert's space-time coordinates [15].

We underlined in red the letter i in front of x_4. Which is not the only thing to be found in this document, which stands as an important document for the history of science and the evolution of scientific ideas. This extract illustrates the way Hilbert represented the world of relativity: with a purely imaginary w4 time coordinate. How could this be possible? Let's not forget that in 1916 nobody imagined that the universe had a history, so much so that when Einstein envisaged his first model he counted on the introduction of the cosmological constant as the keystone of a stationary universe. Quantum mechanics has yet to take off. At most, we know about the electron, identified as an electrically charged particle whose existence has only just been confirmed. The only known forces are electromagnetism and gravity. The article "Fundamentals of Physics" is often described as the first attempt to create a Theory of Everything. Although Hilbert makes no allusion to it, we may well wonder whether behind this attempt was not an attempt to understand the world as a whole, through mathematics.

Hilbert was born into a Protestant family in 1862. He was brought up in a religious school, but when he wrote his articles, at the age of 53, he declared himself an agnostic. He says and writes that science, and mathematics in particular, provides the logical answers, one after the other, in their own time. Engraved on his tombstone is his motto: "Wir müssen wissen, wir werden wissen": "We must know and we will know". In 1916, the universe seemed to be governed by two unique forces: gravitation and electromagnetism. The construction of a field equation using an action seemed to Hilbert to be the ultimate tool, and many authors described his two papers as an "attempt to construct everything", a TOE, theory of everything. Einstein cherished the same dream until his death of a "unified field theory". Today, we know that it's impossible to marry gravitation and electromagnetism without adding an extra dimension (Kaluza space). But Einstein and Hilbert didn't know this. In 1916, we didn't know that the cosmos was evolving, that billions of years earlier it had taken on a very different face. It's hard to imagine that Hilbert, who was unaware of this, wasn't trying to sketch out a scenario for the creation of the universe in his essay. If we decode his approach, the universe is initially a four-dimensional "pseudo-Euclidean" space with coordinates of $\{ w_1, w_2, w_3, w_4 \}$. Everything can be written in this context: the field equation, geodesics, planet trajectories and perhaps electron trajectories. Everything is in place. All that remains is to launch the machine, to create time. Hilbert does this by writing that , which he calls *l* in his reworking of Schwarzschild's solution, is equal to $i\,t$, that this chronological coordinate topples over, revealing its true nature: it is purely imaginary. Let's turn to page 65. The German text shows a four-dimensional "pseudo-Euclidean" space with coordinate.

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Dazu kehren wir wieder zu den ursprünglichen Weltkoordinaten meiner ersten Mitteilung

$$w_1 = x_1, \quad w_2 = x_2, \quad w_3 = x_3, \quad w_4 = ix_4$$

zurück und erteilen den $g_{\mu\nu}$ die entsprechende Bedeutung.

Im Falle der pseudo-Euklidischen Geometrie haben wir

$$g_{\mu\nu} = \delta_{\mu\nu},$$

worin

$$\delta_{\mu\mu} = 1, \qquad \delta_{\mu\nu} = 0 \quad (\mu \neq \nu)$$

bedeutet. Für jede dieser pseudo-Euklidischen Geometrie benachbarte Maßbestimmung gilt der Ansatz

$$(37) \qquad g_{\mu\nu} = \delta_{\mu\nu} + \varepsilon h_{\mu\nu} + \cdots,$$

wo ε eine gegen Null konvergierende Größe und $h_{\mu\nu}$ Funktionen der w_s sind.

Fig.6 : Hilbert [13] page 65.

Translation :

DAVID HILBERT

Pour cela, nous revenons aux coordonnées du monde original de ma première communication

$$w_1 = x_1, \quad w_2 = x_2, \quad w_3 = x_3, \quad w_4 = ix_4,$$

et donnons la signification correspondante aux $g_{\mu\nu}$.

Dans le cas de la géométrie pseudo-euclidienne, nous avons

$$g_{\mu\nu} = \delta_{\mu\nu}$$

où

$$\delta_{\mu\mu} = 1, \qquad \delta_{\mu\nu} = 0 \ (\mu \neq \nu)$$

Pour chaque métrique dans le voisinage de cette géométrie pseudo-euclidienne, l'ansatz

$$g_{\mu\nu} = \delta_{\mu\nu} + \varepsilon h_{\mu\nu} + \cdots$$

est valide, où ε est une quantité tendant vers zéro, et $h_{\mu\nu}$ sont des fonctions de la w_s.

Fig.6bis : Hilbert [13] page 65,

In (37) you have a striking proof of Hilbert's vision of space-time. The zero-order term is the universe before time manifested itself, before the pre-existing geodesic paths that the planets will have to follow when time creates motion. Hilbert calls this original universe pseudo-Euclidean. Its metric tensor corresponds to the Kronecker matrix $\delta_{\mu\nu}$, or unit matrix. And what the universe contains will never be more than a tiny perturbation of a quasi-flat space. In passing, you discover the origin of the signature change, in equation (35):

Die oben genannte geometrische Frage läuft darauf hinaus, zu untersuchen, ob und unter welchen Voraussetzungen die vierdimensionale Euklidische Pseudogeometrie

$$g_{11} = 1, \quad g_{22} = 1, \quad g_{33} = 1, \quad g_{44} = -1$$
$$g_{\mu\nu} = 0 \quad (\mu \neq \nu) \qquad (35)$$

eine Lösung der physikalischen Grundgleichungen bez. die einzige reguläre Lösung derselben ist.

Fig.7 : The origin of the metric change [13]

The geometrical question mentioned above amounts to the investigation, whether and under what conditions the four-dimensional Euclidean pseudo-geometry

$$g_{11} = 1, \quad g_{22} = 1, \quad g_{33} = 1, \quad g_{44} = -1$$
$$g_{\mu\nu} = 0 \quad (\mu \neq \nu) \qquad (35)$$

is a solution, or even the only regular solution, of the basic physical equations.

Fig.7bis : The origin of the metric change

For Hilbert, space is first, time only second. This manifestation of the appearance of time can only be written in the following sequences:

(6)
$$g_{11}\,dw_1^2 + g_{22}\,dw_2^2 + g_{33}\,dw_3^2 + g_{44}\,dw_4^2$$
$$g_{11}\,dx_1^2 + g_{22}\,dx_2^2 + g_{33}\,dx_3^2 - g_{44}\,dx_4^2$$

Before 1939, all mathematicians used the signature $(+---)$, i.e., in particular, they wrote the Lorentz metric

(7)
$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

The fact that velocities cannot exceed c simply translates into the requirement that the length ds be real. This length s is then identified with the proper time τ using the relation $s = c\tau$. In the post-war period, this gradually changed to :

(8) $$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Paradoxically, if the velocity is less than c, the length element becomes purely imaginary and can no longer be identified with the proper time. To obtain the latter, we need to write:

(9) $$d\tau = \frac{1}{c}\sqrt{-(-c^2dt^2 + dx^2 + dy^2 + dz^2)} = \frac{1}{c}\sqrt{-ds^2}$$

Nowhere in the literature can we find an article, or even an argument, justifying this universally practiced generalization of the transition to a signature $(-+++)$ or $(+++-)$. Yet this disconcerting definition of proper time is to be found in Hilbert's 1916 article . It should be noted that the term does not appear at any point in either of his articles. What he is concerned with is a bilinear form :

Wir konstruieren nun in einem jeden Punkte x_1, x_2, x_3 desselben die zu ihm orthogonale geodätische Linie, die eine Zeitlinie sein wird, und tragen auf derselben x_4 als Eigenzeit auf; dem so erhaltenen Punkte der vierdimensionalen Welt weisen wir die Koordinatenwerte $x_1 x_2 x_3 x_4$ zu. Für diese Koordinaten wird, wie leicht zu sehen ist,

(32) $$G(X_s) = \sum_{\mu\nu}^{1,2,3} g_{\mu\nu} X_\mu X_\nu - X_4^2$$

Fig.8 : Hilbert's bilinear form. [13]

I would like to call it; I let x_1, x_2, x_3 be any point coordinates of this space. We now construct at every point x_1, x_2, x_3 of this space the geodesic orthogonal to it, which will be a time line, and on this line we mark off x_4 as proper time; the point in the four-dimensional world so obtained is given coordinate values $x_1 x_2 x_3 x_4$. In these coordinates we have, as is easily seen,

$$G(X_s) = \sum_{\mu\nu}^{1,2,3} g_{\mu\nu} X_\mu X_\nu - X_4^2, \qquad (32)$$

Fig. 8bis: Hilbert's bilinear form.

Hilbert studies the mathematical properties of a bilinear form (32). It doesn't occur to him to describe:

(10) $$s^2 = \sum_{\mu,\nu}^{1,2,3} g_{\mu\nu} X_\mu X_\nu - X_4^{\,2}$$

or, in differential form:

(11) $$ds^2 = \sum_{\mu,\nu}^{1,2,3} g_{\mu\nu}\, dx_\mu dx_\nu - dx_4^{\,2}$$

In fact, we imagine that he doesn't "visualize" this 4D hypersurface at all. He can therefore, without the slightest problem, equip this shape with two different lengths:

nicht sein Vorzeichen ändert: ein Kurvenstück, für welches

$$G\left(\frac{dx_s}{dp}\right) > 0$$

ausfällt, heiße eine *Strecke* und das längs dieses Kurvenstücks genommene Integral

$$\lambda = \int \sqrt{G\left(\frac{dx_s}{dp}\right)}\, dp$$

heiße die *Länge der Strecke*; ein Kurvenstück, für welches

$$G\left(\frac{dx_s}{dp}\right) < 0$$

ausfällt, heiße eine *Zeitlinie* und das längs dieses Kurvenstückes genommene Integral

$$\tau = \int \sqrt{-G\left(\frac{dx_s}{dp}\right)}\, dp$$

heiße die *Eigenzeit der Zeitlinie*; endlich heiße ein Kurvenstück, längs dessen

$$G\left(\frac{dx_s}{dp}\right) = 0$$

wird, eine *Nullinie*.

Fig. 9: The two Hilbert lengths [13]

A piece of the curve for which

$$G(\frac{dx_s}{dp}) > 0$$

shall be called a *segment* and the integral along this piece of curve

$$\lambda = \int \sqrt{G(\frac{dx_s}{dp})}dp$$

shall be the *length of the segment*; a piece of the curve for which

$$G(\frac{dx_s}{dp}) < 0$$

will be called a *time line*, and the integral

$$\tau = \int \sqrt{-G(\frac{dx_s}{dp})}dp$$

evaluated along this piece of curve shall be the *proper time of the time line*; finally a piece of curve along which

$$G(\frac{dx_s}{dp}) = 0$$

shall be called a *null line*.

Fig. 9bis: The two Hilbert lengths

We find the definition of proper time τ. But what is the significance of this second length λ? It evokes some "other physics". The fact that Hilbert's attempt to describe the universe in terms of two different lengths never came to fruition has not attracted the attention of commentators. Does this mean that Hilbert, in his Theory of Everything, envisaged the beginnings of a metaphysics?

Here, then, is mathematician David Hilbert's rather singular conception of the geometry of space-time.

3 – **Comparative constructions of the stationary solution in** SO(3) **symmetry of the Einstein equation by Schwarzschild and Hilbert.**

3a – **The Schwarzschild's calculation:**

We've summarized these two strategies in two full-page illustrations. Let's start with Schwarzschild's construction of the outer metric. Let's start with the first page of his article, which corresponds to issue 189 of the journal in which it is published. We'll refer to this pagination. What is indicated in equation (1) is extremely important.

Über das Gravitationsfeld eines Massenpunktes nach der EINSTEINschen Theorie.

Von K. SCHWARZSCHILD.

(Vorgelegt am 13. Januar 1916 [s. oben S. 42].)

§ 1. Hr. EINSTEIN hat in seiner Arbeit über die Perihelbewegung des Merkur (s. Sitzungsberichte vom 18. November 1915) folgendes Problem gestellt:

Ein Punkt bewege sich gemäß der Forderung

$$\left.\begin{aligned} &\delta\int ds = 0, \\ &\text{wobei} \\ &ds = \sqrt{\sum g_{\mu\nu} dx_\mu dx_\nu} \quad \mu, \nu = 1, 2, 3, 4 \end{aligned}\right\} \quad (1)$$

Fig. 10 : The method followed by Schwarzschild, [49]

Translation :

ON THE GRAVITATIONAL FIELD OF A MASS POINT ACCORDING TO EINSTEIN'S THEORY †

BY K. SCHWARZSCHILD

(Communicated January 13th, 1916

TRANSLATION‡ AND FOREWORD BY

S. Antoci* and A. Loinger'

§1. In his work on the motion of the perihelion of Mercury (see Sitzungsberichte of November 18th, 1915) Mr. Einstein has posed the following problem:

Let a point move according to the prescription:

$$\left\{\begin{aligned} &\delta\int ds = 0, \\ &\textit{where} \\ &ds = \sqrt{\Sigma g_{\mu\nu} dx_\mu dx_\nu} \quad \mu, \nu = 1, 2, 3, 4, \end{aligned}\right. \quad (1)$$

Fig. 10bis : The method followed by Schwarzschild, The first of the two equations indicates that Schwarzschild will minimize the length element s. The second equation states that this length element will necessarily be positive or zero. These two presuppositions will subsequently be of vital importance.

On page 190, he indicates the assumptions that determine his solution, which is supposed to be expressed in a coordinate system $\left\{ x_1, x_2, x_3, x_4 \right\}$. The first coordinate refers to time, the other three to space. The components of the solution metric are independent of time. It then specifies its conditions at infinity. The coefficients of the metric must tend towards those of the Lorentz metric.

(12) $$g_{44} = 1 \quad , \quad g_{11} = -1 \quad , \quad g_{22} = -1 \quad , \quad g_{33} = -1$$

As you can see, he opted for the signature $(+---)$ right from the start, as did Einstein, Droste, Weyl, Flamm and others. But in those days, it never occurred to anyone to do otherwise. On page 191, he accounts for symmetries in his own way, then switches to polar coordinates. His variables are real, so the variable r is necessarily positive or zero.

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$$ds^2 = Fdt^2 - G(dx^2 + dy^2 + dz^2) - H(xdx + ydy + zdz)^2$$

wobei F, G, H Funktionen von $r = \sqrt{x^2+y^2+z^2}$ sind.

Die Forderung (4) verlangt: Für $r = \infty$: $F = G = 1$, $H = 0$.

Wenn man zu Polarkoordinaten gemäß $x = r \sin\vartheta \cos\phi$, $y = r \sin\vartheta \sin\phi$, $z = r\cos\vartheta$ übergeht, lautet dasselbe Linienelement:

$$\begin{aligned} ds^2 &= Fdt^2 - G(dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\phi^2) - Hr^2 dr^2 \\ &= Fdt^2 - (G + Hr^2) dr^2 - Gr^2 (d\vartheta^2 + \sin^2\vartheta d\phi^2). \end{aligned} \quad (6)$$

Fig. 11 : Shift to polar coordinates [49].

§3. If one calls t the time, x, y, z, the rectangular co-ordinates, the most general line element that satisfies the conditions 1-3 is clearly the following:

$$ds^2 = Fdt^2 - G(dx^2 + dy^2 + dz^2) - H(xdx + ydy + zdz)^2$$

where F, G, H are functions of $r = \sqrt{x^2 + y^2 + z^2}$.

The condition (4) requires: for $r = \infty : F = G = 1, H = 0$.

When one goes over to polar co-ordinates according to $x = r\sin\vartheta\cos\phi$, $y = r\sin\vartheta\sin\phi$, $z = r\cos\vartheta$, the same line element reads:

$$\begin{aligned} ds^2 &= Fdt^2 - G(dr^2 + r^2 d\vartheta^2 + r^2 sin^2\vartheta d\phi^2) - Hr^2 dr^2 \\ &= Fdt^2 - (G + Hr^2)dr^2 - Gr^2(d\vartheta^2 + sin^2\vartheta d\phi^2). \end{aligned} \quad (6)$$

Fig. 11bis ; Shift to polar coordinates

Schwarzschild then introduces a change of variables whose sole purpose is to facilitate his calculation of Christoffels nd symbols:

(13) $$x_1 = \frac{r^3}{3} \quad , \quad x_2 = -\cos\vartheta \quad , \quad x_3 = \phi$$

Then he writes :

In den neuen Polarkoordinaten lautet das Linienelement:

$$ds^2 = F dx_4^2 - \left(\frac{G}{r^4} + \frac{H}{r^2}\right) dx_1^2 - G r^2 \left[\frac{dx_2^2}{1 - x_2^2} + dx_3^2 (1 - x_2^2)\right] , \quad (8)$$

wofür wir schreiben wollen:

$$ds^2 = f_4 dx_4^2 - f_1 dx_1^2 - f_2 \frac{dx_2^2}{1 - x_2^2} - f_3 dx_3^2 (1 - x_2^2) . \quad (9)$$

Dann sind $f_1 , f_2 = f_3 , f_4$ drei Funktionen von x_1, welche folgende Bedingungen zu erfüllen haben:

1. Für $x_1 = \infty$: $f_1 = \frac{1}{r^4} = (3x_1)^{-4/3}$, $f_2 = f_3 = r^2 = (3x_1)^{2/3}$, $x_4 = 1$
2. Die Determinantengleichung: $f_1 \cdot f_2 \cdot f_3 \cdot f_4 = 1$.
3. Die Feldgleichungen.
4. Die f stetig, außer für $x_1 = 0$.

Fig. 12 : The Schwarzschild metric in its new coordinates [49]..

Trabslation :

In the new coordinates the line element becomes:

$$ds^2 = F dx_4^2 - \left(\frac{G}{r^4} + \frac{H}{r^2}\right) dx_1^2 - Gr^2 \left(\frac{dx_2^2}{1 - x_2^2} + dx_3^2(1 - x_2^2)\right), \quad (8)$$

Or:

$$ds^2 = f_4 dx_4^2 - f_1 dx_1^2 - f_2 \frac{dx_2^2}{1 - x_2^2} - f_3 dx_3^2(1 - x_2^2), \quad (9)$$

$f_1, f_2, = f_3, f_4$ must fullfil the following conditions:

1. For $x_1 = \infty : f_1 = \frac{1}{r^4} = (3x_1)^{-\frac{4}{3}}, f_2 = f_3 = r^2 = (3x_1)^{\frac{2}{3}}, x_4 = 1$
2. Determinant; $f_1 * f_2 * f_3 * f_4 = 1$
3. Field equations.
4. Continuous f functions outside $x_1 = 0$

Fig. 12 bis : The Schwarzschild metric in its new coordinates

Schwarzschild then calculates the coordinates of the Ricci tensor using Christoffel's symbols. All calculations done, he obtains:

His equation (12) :

(14) $$f_1 = \frac{(3x_1 + \rho)^{-4/3}}{1 - \alpha(3x_1 + \rho)^{-1/3}}$$

His equation (10) :

(15) $$f_2 = (3x_1 + \rho)^{2/3}$$

Page 194 he precises that:

(16) $$f_3 = f_2 = (3x_1 + \rho)^{2/3}$$

His equation (11) :

(17) $$f_4 = 1 - \alpha(3x_1 + \rho)^{-1/3}$$

In his equation (13) he precises that $\rho = \alpha^3$. Replacing x_1 by $r^3/3$ gives :

(18)

$$ds^2 = \frac{(r^3 + \alpha^3)^{1/3} - \alpha}{(r^3 + \alpha^3)^{1/3}} c^2 dt^2 - \frac{r^4}{(r^3 + \alpha^3)\left[(r^3 + \alpha^3)^{1/3} - \alpha\right]} dr^2 - (r^3 + \alpha^3)^{2/3}(d\theta^2 + \sin^2\theta d\varphi^2)$$

α is what will later be called the Schwarzschild length. No one has ever explained what is the true expression of the original solution found by Schwarzschild in January 1916, 108 years ago, expressed using the coordinates he defined at the beginning of his article:

(19) $$\{t, x, y, z,\} \rightarrow \left\{t, r = \sqrt{x^2 + y^2 + z^2}, \vartheta, \phi,\right\}$$

Nor has anyone examined its properties for over a century. Let's do it. As r tends to zero, g_{tt} tends to zero. The term g_{rr} gives the indeterminate form $\frac{0}{0}$,and the terms $g_{\vartheta\vartheta}$ and $g_{\phi\phi}$ give:

(20) $$f_2 \rightarrow (r^2 + \alpha^3)^{2/3} \qquad f_3 \rightarrow (r^2 + \alpha^3)^{2/3} \sin^2\vartheta$$

The non-contractile nature of the 4-dimensional hypersurface comes to the fore. When, at t = Cst , r = Cst , $\vartheta = \pi / 2$, we calculate the length :

(21) $$p = \int_0^{2\pi} (r^2 + \alpha^3)^{1/3} d\phi = 2\pi (r^2 + \alpha^3)^{1/3}$$

this perimeter has, for r = 0, the minimal value $2\pi\alpha$. We now turn to David Hilbert's article [13]:

3a – **Hilbert calculus:**

On page 67 of the original article, he begins by setting out his hypotheses:

Die Annahmen über die $g_{\mu\nu}$ sind folgende:

1. Die Maßbestimmung ist auf ein Gaußisches Koordinatensystem bezogen — nur daß g_{44} noch willkürlich gelassen wird; d. h. es ist

$$g_{14} = 0, \quad g_{24} = 0, \quad g_{34} = 0.$$

2. Die $g_{\mu\nu}$ sind von der Zeitkoordinate x_4 unabhängig.
3. Die Gravitation $g_{\mu\nu}$ ist zentrisch symmetrisch in Bezug auf den Koordinatenanfangspunkt.

Nach Schwarzschild ist die allgemeinste diesen Annahmen entsprechende Maßbestimmung in räumlichen Polarkoordinaten, wenn

$$w_1 = r\cos\vartheta$$
$$w_2 = r\sin\vartheta\cos\varphi$$
$$w_3 = r\sin\vartheta\sin\varphi$$
$$w_4 = l$$

gesetzt wird, durch den Ausdruck

(42) $$F(r)\,dr^2 + G(r)\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) + H(r)\,dl^2$$

Fig.13 : Hilbert's assumptions [13]

Translation :

gations. The assumptions about the $g_{\mu\nu}$ are the following:

1. The metric is represented in a Gaussian coordinate system, except that g_{44} is left arbitrary, i.e. we have

$$g_{14} = 0, \qquad g_{24} = 0, \qquad g_{34} = 0.$$

2. The $g_{\mu\nu}$ are independent of the time coordinate x_4.
3. The gravitation $g_{\mu\nu}$ is centrally symmetric with respect to the origin of coordinates.

According to Schwarzschild the most general metric conforming to these assumptions is represented in polar coordinates, where

$$w_1 = r\cos\vartheta$$
$$w_2 = r\sin\vartheta\cos\varphi$$
$$w_3 = r\sin\vartheta\sin\varphi$$
$$w_4 = l,$$

by the expression

$$F(r)dr^2 + G(r)(d\vartheta^2 + \sin^2\vartheta d\varphi^2) + H(r)dl^2 \qquad (42)$$

where $F(r), G(r), H(r)$ are still arbitrary functions of r. If we put

Fig.13 bis : Hilbert's assumptions

- This expression of the metric (like the one Schwarzschild opted for) is not the most general, given the initial assumptions: independence with respect to time, spherical symmetry, but we'll see that later.

- Hilbert resumes his vision of a universe with a signature metric

  His formulation (42) of the bilinear form is not exactly that of Schwarzschild. See Figures 12 and 12 bis and his equations (6). Its factor coefficient is not G but Gr2. Consider this as a detail.

With this, Hilbert writes:

(42) $$F(r)\,dr^2 + G(r)\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) + H(r)\,dl^2$$

dargestellt, wo $F(r)$, $G(r)$, $H(r)$ noch willkürliche Funktionen von r sind. Setzen wir

$$r^* = \sqrt{G(r)},$$

so sind wir in gleicher Weise berechtigt r^*, ϑ, φ als räumliche Polarkoordinaten zu deuten. Führen wir in (42) r^* anstatt r ein und lassen dann wieder das Zeichen $*$ weg, so entsteht der Ausdruck

(43) $$M(r)\,dr^2 + r^2\,d\vartheta^2 + r^2\sin^2\vartheta\,d\varphi^2 + W(r)\,dl^2,$$

wo $M(r)$, $W(r)$ die zwei wesentlichen willkürlichen Funktionen von r bedeuten. Die Frage ist, ob und wie diese auf die allgemeinste Weise zu bestimmen sind, damit den Differentialgleichungen (36) Genüge geschieht.

Fig. 14 : His so called "radial" variable r* is no longer $\sqrt{x^2+y^2+z^2}$. [13]

Translation:

where $F(r)$, $G(r)$, $H(r)$ are still arbitrary functions of r. If we put

$$r^* = \sqrt{G(r)},$$ ← why not

then we are equally justified in interpreting r^*, ϑ, φ as spatial polar coordinates. If we introduce r^* in (42) instead of r and then eliminate the sign $*$, the result is the expression

$$M(r)dr^2 + r^2 d\vartheta^2 + r^2\sin^2\vartheta d\varphi^2 + W(r)dl^2, \qquad (43)$$ ← NO !

where $M(r)$, $W(r)$ mean the two essential, arbitrary functions of r. The question is whether and how these can be determined in the most general way so that the differential equations (36) enjoy satisfaction.

Fig. 14 bis : His so called "radial" variable r* is no longer $\sqrt{x^2+y^2+z^2}$.

It is perfectly legal to make this change of variable $r^* = \sqrt{G(r)}$, which only requires the determination of two unknown functions $M(r^*)$ and $W(r^*)$. but we must then write the bilinear form (43) in the form :

(22) $$M(r^*)dr^2 + r^{*2}(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) + W(r^*)dt^2$$

What is this equation (36) whose solution we are looking for? It is:

(23) $$K_{\mu\nu} - \frac{1}{2} K g_{\mu\nu} = 0 \qquad (36)$$

As we know that the letter K refers to the Ricci tensor and scalar, this is the Einstein equation without a second member. The rest of the calculation can then be repeated, up to the result, provided it is expressed using the variable r* and not r :

und, wenn wir

$$M = \frac{r^*}{r^* - m} \qquad W = w^2 \frac{r^* - m}{r^*}$$

setzen, wo nunmehr m und w die unbekannten Funktionen von r^*

Fig. 15: The Hilbert's functions *M* and *N*. [13].

Translation :

and if we put

$$M = \frac{r^*}{r^* - m} \qquad W = w^2 \frac{r^* - m}{r^*}$$

where now m and w are the unknown functions of r^* we finally obtain

Fig. 15 bis : The Hilbert's functions *M* and *N*.

ilbert then expresses his result, i.e. the desired bilinear form:

den gemachten Annahmen 1., 2., 3., dar. Nehmen wir als Integrale von (44) $m = \alpha$, wo α eine Konstante ist und $w = 1$, was offenbar keine wesentliche Einschränkung bedeutet, so ergibt sich aus (43) für $l = it$ die gesuchte Maßbestimmung in der von Schwarzschild zuerst gefundenen Gestalt

(45) $$G(dr, d\vartheta, d\varphi, dl) = \frac{r}{r-\alpha} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta \, d\varphi^2 - \frac{r-\alpha}{r} dt^2.$$

Die Singularität dieser Maßbestimmung bei $r = 0$ fällt nur dann fort, wenn $\alpha = 0$ genommen wird, d. h. Die Maßbestimmung der pseudo-Euklidischen Geometrie ist bei den An-

Fig. 16 : The result of Hilbert's calculus. [13].

Translation :

> It is easy to convince oneself that these equations indeed imply that all $K_{\mu\nu}$ vanish; they therefore represent essentially the most general solution of equations (36) under the assumptions 1., 2., 3., we made. If we take as integrals of (44) $m = \alpha$, where α is a constant, and $w = 1$, which evidently is no essential restriction, then for $l = it$ (43) results in the desired metric in the form first found by Schwarzschild
>
> $$G(dr, d\vartheta, d\varphi, dl) = \frac{r}{r-\alpha}dr^2 + r^2 d\vartheta^2 + r^2 \sin^2\vartheta d\varphi^2 - \frac{r-\alpha}{r}dl^2. \qquad (45)$$
>
> The *singularity* of the metric at $r = 0$ disappears only if we take $\alpha = 0$

Fig. 16 bis : The result of Hilbert's calculus

In his paper above, Hilbert shows that his quantities m and w, which in no way represent the modulus of the vector (w_1, w_2, w_3, w_4), are constants. The constant *m* is then identified with α, the Schwazrschild length, and *w* with the unit.

As a simple remark, using an elliptical or hyperbolic metric would not change the projection of a geodesic trajectory onto the plane. By ignoring time, i.e. by keeping this variable l Hilbert could just as well have presented his result, provided he kept his variable r* according to

As a simple remark, using an elliptical or hyperbolic metric would not change the projection of a geodesic trajectory onto the plane. By ignoring time, i.e. by keeping this variable l Hilbert could just as well have presented his results, provided he kept his variable r* according to:

$$(24) \qquad G(dr^*, d\vartheta, d\phi, dl) = \frac{r^*}{r^*-\alpha}dr^{*2} + r^{*2} d\vartheta^2 + r^{*2}\sin^2\vartheta d\phi^2 + \frac{r^*-\alpha}{r^*}dl^2$$

In equation (45), we find a typographical error. This result is in fact:

$$(25) \qquad G(dr^*, d\vartheta, d\phi, dt) = \frac{r^*}{r^*-\alpha}dr^{*2} + r^{*2} d\vartheta^2 + r^{*2}\sin^2\vartheta d\phi^2 - \frac{r^*-\alpha}{r^*}dt^2$$

→ Hilbert's mistake is to confuse his bilear form with « the form fist found by Schwarzschild » (« der von Schwarzschild zuerst gefundenen Gestalt ». He does not percieve the fundamental difference between his « radial » coordinate r and Schwarzschild's intermediate variable R. His confusion can be seen in footnote 7, where he writes : « To transforme the location at R = α , as Schwarzschild does (...)

, is note recommanded in my opinion ; Schwarzschild's transformation is moreover not the simpliest that achieves rhis goal ».

So, Hilbert thinks his own radial is the right one, and Schwarzschild only involved a change of coordinates to place « the singularity » (but in fact is not a true singularity) towards the origin.

Such confusion, taken up by his successors, will have incalculable consequences. It's an error in the sense that Hilbert mentions the existence of a singularity ("in r = 0").

Note that J.Droste [5], who is taken as a reference by C.Corda [4], commits the same confusion. He begins by introducing the form of a metric depending on three unknown functions, in his equation (2):

For a centre at rest and symmetrical in all directions it is easily seen that

$$ds^2 = w^2\, dt^2 - u^2\, dr^2 - v^2\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2), \quad . \quad . \quad (2)$$

w, u, v only depending on r, and (ϑ, φ) representing polar coordinates. Now, if g_{ij} and therefore also g^{ij} are all zero, if $i =/= j$, G

Fig. 17 : Droste, the initial form of his metric [5]

But, like Hilbert, he soon reduced the number of unknown functions to two:

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The equations of the field being covariant for all transformations of the coordinates whatever, we are at liberty to choose instead of r a new variable which will be such a function of r, that in ds^2 the coefficient of the square of its differential becomes unity. That new variable we name r again and we put

$$ds^2 = w^2\, dt^2 - dr^2 - v^2\,(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2) \quad . \quad . \quad . \quad . \quad (4)$$

w and v only depending on r. We now find

Fig. 18: Droste [5] makes the same mistake as Hilbert.

Note that Droste opts for the signature $(+---)$. A at the end of this calculation, using several successive changes of variables he produces, at the end of a final change, what Corda calls the "Standard Schwarzschild solution":

$$ds^2 = (1-\xi)\,dt^2 - \frac{4\alpha^2}{(1-\xi)\,\xi^4}\,d\xi^2 - \frac{\alpha^2}{\xi^2}\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right).$$

Lastly we put

$$\xi = \frac{\alpha}{r}$$

This r is not the same as occurs in (4). We obtain

$$ds^2 = \left(1-\frac{\alpha}{r}\right)dt^2 - \frac{dr^2}{1-\frac{\alpha}{r}} - r^2\left(d\vartheta^2 + \sin^2\vartheta\, d\varphi^2\right) \qquad (7)$$

Fig. 19 : Final result [5].

And, finally, this new coordinate, which he always calls r, gives a quadratic form identical to Hilbert's. But it's by no means the initial radial coordinate. But this is by no means the original radial coordinate. Droste, unlike Hilbert, shows a certain caution, which Hilbert did not, envisioning from the outset that a singularity could correspond to the zero value of his variable r. Here's the passage from Droste's article :

3. From (7) we can immediately deduce some conclusions. The point (r, ϑ, φ) lies at a distance

$$\delta = \int_{\alpha}^{r} \frac{dr}{\sqrt{1-\frac{\alpha}{r}}} = r\sqrt{1-\frac{\alpha}{r}} + \alpha \log\left(\sqrt{\frac{r}{\alpha}-1}+\sqrt{\frac{r}{\alpha}}\right). \quad (8)$$

[1]) After the communication to the Academy of my calculations, I discovered that also K. SCHWARZSCHILD has calculated the field. Vid : Sitzungsberichte der der Kon. Preuss. Akad. der Wiss. 1916, page 189. Equation (7) agrees with (14) there, if R is read instead of r.

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from the point, where the radius intersects sphere $r = \alpha$, if $r > \alpha$ and supposing that (7) remains valid up to $r = \alpha$. In future we will always make these two suppositions, as we shall see, that a moving particle outside sphere $r = \alpha$ can never pass that sphere, we may, in studying its motion. disregard the space $r < \alpha$. Should (7) cease to be valid as soon as r becomes $< R$, we need only exclude the space $r < R$ from the conclusions which will still be made, to make them valid again.

If r be very large with respect to α, the proportion $\delta : r$ approaches to 1.

The circumference of a circle $r = const.$ is $2\pi r$ by (7); this shows how r can be measured. Circle α has the circumference $2\pi\alpha$.

One might in (7) perform a substitution $t = f(r,\tau)$. Then a term containing $drd\tau$ would arise and the velocity c of light, travelling along r, would have to be calculated from an equation of the form

$$F_1(r,\tau) + F_2(r,\tau)\,c - F_3(r,t)\,c^2 = 0$$

and would have *two* values, one for light coming from the centre, the other for light moving towards it. Moreover these values would depend on t. In consequence of the last fact we should not name

Fig. 20 Droste, pages 200-201 [5]

These are only the conclusions of a theorist for whom everything must be real, including the element of length ds. This implies that $r \geq \alpha$ (i.e. in Schwarzschild notation : $R \geq \alpha$). The portions of space for which $r < \alpha$ (i.e. $R < \alpha$) are excluded from the solution.

Droste then remarks on the possibility of a cross term in dr dt. But the way he introduces it leads him to conclude that such a term would depend on the time coordinate, which is by no means an obligation. More on this later.

3b – **Hilbert's exploitation of his solution.**

Constructing a solution to Einstein's equation means first and foremost being able to produce geodesics. Hilbert:

Die Differentialgleichungen der geodätischen Linien für das zentrische Gravitationsfeld (45) entspringen aus dem Variationsproblem

$$\delta\int\left(\frac{r}{r-\alpha}\left(\frac{dr}{dp}\right)^2+r^2\left(\frac{d\vartheta}{dp}\right)^2+r^2\sin^2\vartheta\left(\frac{d\varphi}{dp}\right)^2-\frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2\right)dp=0,$$

Fig. 21 : Hilbert minimizes the square of the length. [13]

The differential equations of geodesic lines for the centrally symmetric gravitational field (45) arise from the variational problem

$$\delta\int\left(\frac{r}{r-\alpha}\left(\frac{dr}{dp}\right)^2+r^2\left(\frac{d\vartheta}{dp}\right)^2+r^2\sin^2\vartheta\left(\frac{d\varphi}{dp}\right)^2-\frac{r-\alpha}{r}\left(\frac{dt}{dp}\right)^2\right)dp=0,$$

Fig. 21 bis : Hilbert minimizes the square of the length

This approach would be adopted by many of his successors over the following century. It should be noted that Schwarzcshild, for his part, does not write this variation of the action integral, as does Droste:

4. We now proceed to the calculation of the equations of motion of a particle in the field.

The equations of motion express the fact that the first variation of the integral

$$\int_{t_1}^{t_2} L\, dt$$

will be zero, if the varied positions for $t = t_1$ and $t = t_2$ are the same as the actual ones. L represents the quantity

$$L = \frac{ds}{dt} = \sqrt{1 - \frac{\alpha}{r} - \frac{\dot{r}^2}{1 - \frac{\alpha}{r}} - r^2 \dot{\vartheta}^2 - r^2 \sin^2 \vartheta\, \dot{\varphi}^2}, \quad . \quad . \quad (9)$$

Fig. 22: Droste, his Lagrangian. [5]

Translation :

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$$L = \frac{ds}{dt} = \sqrt{1 - \frac{\alpha}{r} - \frac{\dot{r}^2}{1 - \frac{\alpha}{r}} - r^2 \dot{\vartheta}^2 - r^2 \sin^2 \vartheta\, \dot{\varphi}^2}, \quad . \quad . \quad (9)$$

Fig. 22 bis ; Droste, his Lagrangian.

$$\delta\int\left(\frac{R}{R-\alpha}\left(\frac{dR}{dp}\right)^2+R^2\left(\frac{d\vartheta}{dp}\right)^2-R^2\sin^2\vartheta\left(\frac{d\phi}{dp}\right)^2-\frac{R-\alpha}{R}\left(\frac{dt}{dp}\right)^2\right)dp=0$$

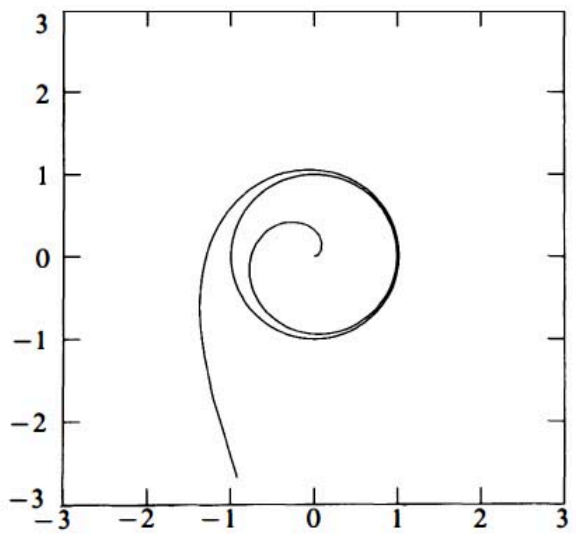
$$\downarrow$$

Lagrange eq.

$$\uparrow$$

$$\delta\int\sqrt{\frac{R-\alpha}{R}\left(\frac{dt}{dp}\right)^2-\frac{R}{R-\alpha}\left(\frac{dR}{dp}\right)^2-R^2\left(\frac{d\vartheta}{dp}\right)^2-R^2\sin^2\vartheta\left(\frac{d\phi}{dp}\right)^2}\,dp=0$$

Both approaches lead to the same system of Lagrange equations. These produce geodesic curves in (R,ϕ), complete in the first case, and interrupted when $R<0$ in the second. With a coordinate singularity that can be eliminated. The corresponding curves are given below



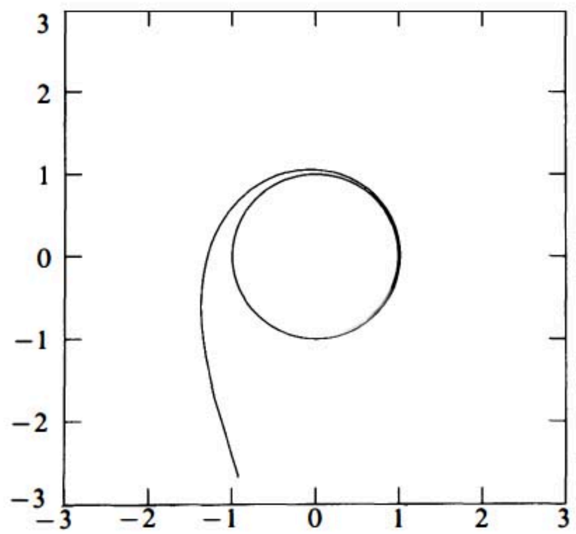


Fig.23 : Plunging geodesics. On the left, the particle falls towards the center as it spirals. On the right, its trajectory is interrupted when it reaches the Schwarzschild sphere [50].

The part of the curve corresponding to $R<\alpha$, which is algebraically real, does not enter the realm of physics if we assume that the length measurement along this part of the curve must be real. Before going any further, we should mention the contributions of L.Flamm and H.Weyl.

As we shall see later, Karl Schwarzschild, who was both an excellent mathematician-geometerist and a seasoned physicist, followed a physicist's logic. The geometric description of the gravitational field created by a mass corresponding to a sphere filled with incompressible material could only be achieved by connecting two metrics, the first describing the geometry outside the mass and the second inside it..

5 – **Richard Tolman's contribution** [18].

These elements, which were to form the basis of a scientific cosmology, were born in Germany and Austria before the Second World War. These founding texts were originally written in German, and at the time, the dissemination of ideas was achieved through the sending of offprints, a very small number of printed copies of articles. There were, of course, direct contacts, when authors of works came to lecture outside their home country, as was the case when Einstein gave a series of lectures at Columbia University and Princeton in 1921. The first to bring this body of knowledge together, in English, was mathematician Richard Tolman, in the form of a book[18] published in 1934, which quickly gained a worldwide following. The only solution evoked was that describing the exterior of a mass. The metric supposed to account for symmetries is as follows:

In accordance with the static and spherically symmetrical nature of the field which would surround an attracting point particle, it can be shown necessarily possible (see § 95) to choose coordinates r, θ, ϕ, and t such that the line element will be of the simple form

$$ds^2 = -e^\lambda\, dr^2 - r^2\, d\theta^2 - r^2 \sin^2\theta\, d\phi^2 + e^\nu\, dt^2, \qquad (82.1)$$

where λ and ν are functions of r alone. Furthermore, the components

Fig. 24 : The Tolman metric, with its two exponential functions [18]..

The functions e^λ and e^ν are introduced in such a way as to ensure the invariance of the signature of the metric, which retains its usual formulation $(+---)$. Note the qualifier "static", which is not stationary. By stationary we mean a solution that is independent of time. By static, we mean a solution that is both time-independent and symmetrical when t is changed to -t. There is no physical imperative to opt for this symmetry. Note, however, that this excludes any dr dt cross term. The author then gives the result of the calculation, identical to Droste's formula, taking into account - of little interest - the presence of the cosmological constant in the field equation

$$ds^2 = -\frac{dr^2}{1-\frac{2m}{r}-\frac{\Lambda r^2}{3}} - r^2\, d\theta^2 - r^2\sin^2\theta\, d\phi^2 + \left(1-\frac{2m}{r}-\frac{\Lambda r^2}{3}\right) dt^2, \qquad (96.3)$$

Fig. 25 : Tolman's external metric [18] .

Tolman read and spoke fluent German, which enabled him to read Schwarzchild's second paper, from February 1916, describing the geometry inside a mass. More on this later. But note that he makes no mention of any restrictive conditions concerning the value of the r coordinate present in his equation (96.3).

5 – **Back to geometric considerations**.

In February 1916, K.Schwarzschild completed his January paper by publishing a second one in which he constructed a stationary, spherically symmetric solution to Einstein's second-member equation, describing the geometry of the portion of space corresponding to the interior of a sphere filled with an incompressible material. In terms of spatial coordinates, he uses two angles: ϕ and χ, which identify the position of a point in the sphere. The value $\chi = 0$ corresponds to the object's center and $\chi = \chi_a$ to its outer surface. The radial coordinate is then :

(27) $$R = \hat{R}\cos\chi$$

And this second characteristic length is:

(28) $$\hat{R} = \sqrt{\frac{3c^2}{8\pi G\rho_o}}$$

This internal metric is written as:

(29) $$ds^2 = \left(\frac{3\cos\chi_a - \cos\chi}{2}\right)^2 c^2dt^2 - \frac{3c^2}{8\pi G\rho_o}(d\chi^2 + \sin^2\chi d\theta^2 + \sin^2\chi\sin^2\theta d\theta^2)$$

We can then use this "intermediate quantity" R (« Hilsgröβe ») from Schwarzschild to describe the geometry outside the mass. If M is the mass of the object, this metric can be written as :

(30) $$ds^2 = \left(1 - \frac{2GM}{c^2R}\right)c^2dt^2 - \frac{dR^2}{1 - \frac{2GM}{c^2R}} - R^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

In 1917, F.Flamm perfectly described such an approach by cutting the four-dimensional hypersurface to reveal its meridian. See figure 2. A form of geometric criticality arises when we consider an object of constant density, whose mass would increase progressively. If the density is assumed the density to be constant, so $\hat{R}$ is the characteristic radius. The Schwarzschild radius increases as the mass increases, and therefore as the cube of the radius. For a star like the Sun, the radius is of the order of 3 kilometers, whereas the radius is a hundred times the star's radius. We thus come up against a first criticality, which we'll call geometric, when the Schwarzschild radius joins this radius $\hat{R}$. If we trace the meridian of the hypersurface, it corresponds to a half-circle connecting with a lying half-parabola:

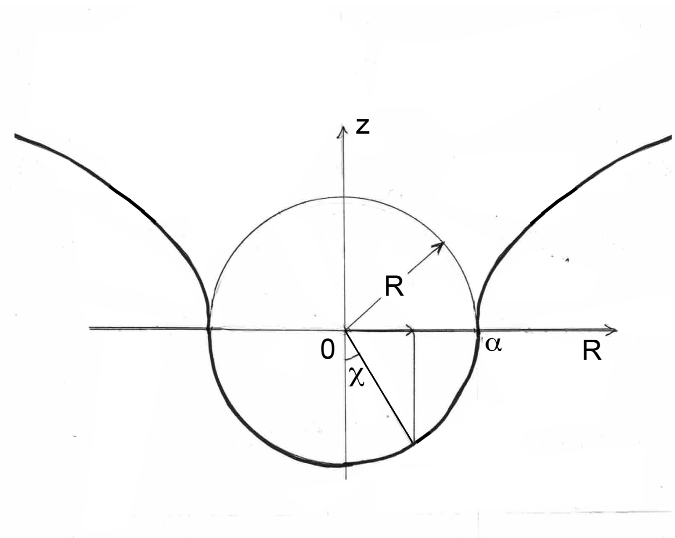


Fig. 26: First form of geometric criticality.

The question then arises: what happens when the influx of matter overtakes this situation and the description provided by K. Schwarzschild, using two connecting metrics, is no longer adequate?

6 – **Topological extensions**.

These were envisaged even before it was thought that such stars could exist, in the writings of those who attempted to represent masses using topology. Weyl is a pioneer in this field [6]. His representation has already been mentioned in Section 1 and in the extract from Fig. 1, but he goes further by proposing a new change of variable:

lungen willen transformieren muß. Die Transformationsformeln sollen lauten

$$x_1' = \frac{r'}{r} x_1, \quad x_2' = \frac{r'}{r} x_2, \quad x_3' = \frac{r'}{r} x_3; \quad r = \left(r' + \frac{a}{2}\right)^2 \cdot \frac{1}{r'} \cdot$$

Lasse ich nach Durchführung der Transformation die Akzente wieder fort, so ergibt sich

$$(12) \quad d\sigma^2 = \left(1 + \frac{a}{2r}\right)^4 (dx_1^2 + dx_2^2 + dx_3^2), \quad f = \left(\frac{r - a/2}{r + a/2}\right)^2 \cdot$$

In den neuen Koordinaten ist das Linienelement des Gravitationsraumes also dem Euklidischen *konform*; das lineare Vergrößerungsverhältnis ist

$$\left(1 + \frac{a}{2r}\right)^2 \cdot$$

$d\sigma^2$ ist regulär für alle Werte $r > 0$, f ist durchweg positiv und wird nur für

$$r = \frac{a}{2}$$

zu Null. Der Umfang des Kreises $x_1^2 + x_2^2 = r^2$ beträgt

$$2\pi r \left(1 + \frac{a}{2r}\right)^2;$$

Fig.27 : Weyl isotropic coordinates. [6]

This may become clearer upon introducing another coördinate system into which I need to transform Schwarzschild's equations at any rate in order to proceed further. The transformation equations read

$$x_1{}' = \frac{r'}{r}x_1, \quad x_2{}' = \frac{r'}{r}x_2 \quad x_3{}' = \frac{r'}{r}x_3; \qquad r = \left(r' + \frac{a}{2}\right)^2 \cdot \frac{1}{r'}.$$

If I remove the primes after carrying out the transformation, then

$$(12) \qquad d\sigma^2 = \left(1 + \frac{a}{2r}\right)^4 (dx_1{}^2 + dx_2{}^2 + dx_3{}^2), \qquad f = \left(\frac{r - a/2}{r + a/2}\right)^2$$

results. In the new coördinates, the line element of the gravitational space is thus *conformal* to Euclidean space; the linear enlargement factor is

$$\left(1 + \frac{a}{2r}\right)^2.$$

$d\sigma^2$ is regular for all values $r > 0$, f is always positive and becomes zero only for

$$r = \frac{a}{2}.$$

The circumference of the circle $x_1{}^2 + x_2{}^2 = r^2$ is

$$2\pi r \left(1 + \frac{a}{2r}\right)^2;$$

if we allow r to run over its range of values beginning with $+\infty$, then this function decreases monotonically until it reaches the value $4\pi a$ for

$$r = \frac{a}{2},$$

Fig.27 bis : Weyl isotropic coordinates.

Replacing r' with r is not desirable. Coordinates are only ever an attempt to read geometry. It would have been preferable to use another letter. Thus, given that Weyl starts from:

$$(31) \qquad ds^2 = f\, dt^2 - d\sigma^2 \geq 0$$

$d\sigma^2$ designating the spatial part of the metric, we should write :

$$(32) \qquad d\sigma^2 = \left(1 + \frac{a}{2u}\right)^4 (\, dx_1{}^2 + dx_2{}^2 + dx_3{}^2\,)$$

(33)
$$f = \left(\frac{u - a/2}{u + a/2} \right)^2$$

He rediscovers the non-contractile nature of the object, i.e. the perimeter of a centered closed curve has a minimum value $p = 4\pi a = 2\pi\alpha$. But he goes further, expressing the proper time according to :

(34)
$$ds = \sqrt{f}\ dt = \left(\frac{u - a/2}{u + a/2} \right) dt$$

The second layer of the hypersurface is traversed for values of a ranging from a/2 (throat sphere) to zero. The above factor then becomes negative. The length element ds cannot become negative, which means that the proper time cannot be reversed. So, to maintain the sign of ds, the time coordinate must reverse along this second layer, as Weyl explicitly states:

dem Innern des Massenpunktes entsprechen. Bei analytischer Fortsetzung wird

$$\sqrt{f} = \frac{r - a/2}{r + a/2}$$

im Innern negativ, so daß also dort für einen ruhenden Punkt kosmische Zeit (x_4) und Eigenzeit gegenläufig sind. (In der

Fig. 28 : Weyl: time coordinate inversion. [6]

to the inside of the point mass. When continued analytically,

$$\sqrt{f} = \frac{r - a/2}{r + a/2}$$

becomes negative in the inside region, meaning that for a point at rest, the cosmic time (x_4) and proper time run in opposite directions. (In Nature, it

Fig. 28 bis : Weyl: time coordinate inversion..

Here, Weyl, who is still trying to give a topological interpretation to masses, speaks of the "interior" of such an object, which shows that he is also falling into the trap of replacing r' with r', in the belief of constructing a new radial coordinate, going from zero to infinity. If we return this magnitude to its character as a simple parameter, focusing our attention solely on the intrinsic magnitude, the length s, it's clear that when u tends towards zero, the perimeter p tends towards infinity.

→ Weyl was the first, in 1916, to envisage that the geometry associated with the Schwarzschild outer metric, considered in isolation, translates a bridge betwee two T-symmetric spacetimes

This representation of masses as topological singularities is also the basis of Einstein and Rosen's 1935 paper [20]. Note, however, that this extension is not Lorentzian to infinity.

Another type of extension [10] uses a change of variable:

(35) $$R = \alpha (1 + L_n \rho)$$

The metric then becomes:

(36) $$ds^2 = \frac{L_n \mathrm{ch}\rho}{1 + L_n \mathrm{ch}\rho} c^2 dt^2 - \alpha^2 \frac{1 + L_n \mathrm{ch}\rho}{L_n \mathrm{ch}\rho} \mathrm{th}^2\rho \, d\rho^2 - \alpha^2 \left(1 + L_n \mathrm{ch}\rho \right)^2 \left(d\theta^2 + \sin^2\theta \, d\varphi^2 \right)$$

The two layers are then traversed by varying the coordinate ρ from $-\infty$ to $+\infty$, a throat sphere corresponding to the value $\rho = 0$. On this throat sphere, the term g_{tt} tends towards zero and the term $g_{\rho\rho}$ becomes $\frac{0}{0}$. By moving ρ towards this zero value, a limited development shows that $g_{\rho\rho}$ tends towards 2, which is another way of showing that on the throat sphere we're simply dealing with a singularity of coordinate s.

7 – **The questionable use of a stationary solution to describe a highly unsteady process.**

The existence of a new particle, the neutron, became clear in 1932, just as Tolman was writing his book. This was a time when the development of theoretical models and the influx of experimental and observational results mirrored each other. With the advent of quantum mechanics, nuclear physics and an understanding of the mechanics at work within stars, the idea of the instability of massive stars, at the end of their lifetimes, was gaining ground. In 1939, R. Oppenheimer and Snyder [19] proposed using the external metric, considered in isolation, to describe an object undergoing implosion. The starting point is the time it takes for a witness mass, in free fall, to reach the Sshwarzschild sphere. If the proper time is finite and very short, then by choosing a t coordinate in spherical symmetry and under “static” conditions, this time becomes infinite.

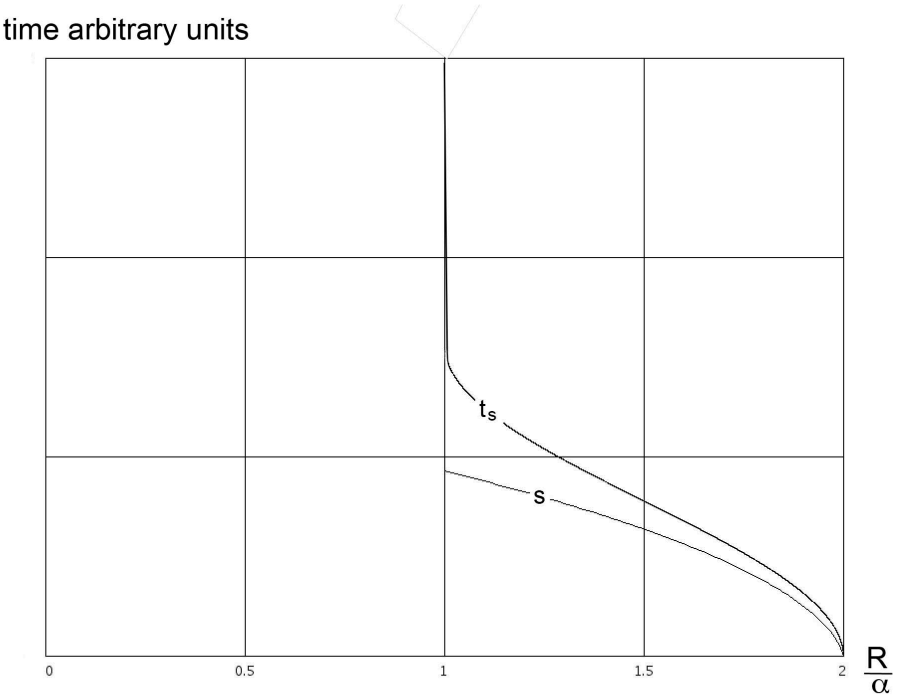


Fig. 29 : Free-fall time soars as soon as
the particle approaches the Schwarzschild sphere.

This aspect is the basis of a proposed model for the implosion of a massive star. If all these massive particles implode without the force of pressure being able to oppose them, and if collapse, this implosion, although taking place in its own time in a duration measured in seconds, seems to last an infinite time for an outside observer, then the outside metric may suffice to describe this phenomenon. In 1939, through this study, Oppenheimer and Snyder [20] gave birth to a new object, to which John Archibald Wheeler gave the name black hole.

This scenario no longer holds if we take into account a dr dt cross term in the metric. In 1924, A. Eddington, followed in 1958 by D. Finkelstein [17], introduced such a term by means of a simple change of variable affecting the time variable, :

(37) $$c\,t_E = c\,t_S + \delta\,\alpha\,L_n\left(\frac{R}{\alpha} - 1\right) \qquad \delta = \pm 1$$

t_S designating "Schwarzschild time"). The metric becomes, in R with $R \le \alpha$:

(38) $$ds^2 = \left(1 - \frac{\alpha}{R}\right)c^2 dt_E{}^2 - \left(1 + \frac{\alpha}{R}\right)dR^2 - R^2(\,d\theta^2 + \sin^2\theta\, d\varphi^2\,) + \frac{2\delta\alpha c}{R}dR\,dt_E$$

With the space coordinate ρ :

(39)

$$ds^2 = \frac{L_n\,ch\rho}{1 + L_n\,ch\rho}c^2 dt_E^2 - \alpha^2\frac{L_n\,ch\rho}{1 + L_n\,ch\rho}\,th^2\rho\, d\rho^2 + \frac{2\delta c\alpha}{\rho}d\rho\, dt_E - \alpha^2\left(\,1 + L_n\,ch\rho\,\right)^2\left(\,d\theta^2 + \sin^2\theta\, d\varphi^2\,\right)$$

Eddington had imagined this change of coordinate only to transform the situation in R $=\alpha$ into a simple coordinate singularity problem. But in 2021, the mathematician P. Koiran [21] showed that a dissymmetry between the free-fall and escape times in the two slicks would then become apparent. If we opt for $\delta = -1$ in the portion of space-time associated with the ordinary world and for $\delta = +1$ I the second fold, we have :

- a brief, finite free-fall time and an infinite escape time in the first sheet

- A finite, brief escape time and an infinite free-fall time in the second sheet.

This gives the structure the character of a "one-way membrane". This situation invalidates the hypothesis of an infinite free-fall time in the first sheet, and thus invalidates the black hole model, which relies on a "freeze-frame".

Furthermore, taking up Weyl's work on the inversion of the time coordinate at the passage of the gorge sphere, and relying on the theorem of mathematician JM. Souriau [22], equation (14.67), the inversion of the time coordinate (T-symmetry) goes hand in hand with the inversion of energy and mass. Energy being the source of the gravitational field, it is then possible to describe this field in a "positive world", using a mass M :

$$(40) \quad ds^2 = \left(1 - \frac{2GM}{c^2 R}\right) c^2 dt_E{}^2 - \left(1 + \frac{2GM}{c^2 R}\right) dR^2 - R^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) - \frac{4GM}{cR} dR \, dt_E$$

And on the other fold, in the negative world:

$$(41) \quad ds^2 = \left(1 + \frac{2GM}{c^2 R}\right) c^2 dt_E{}^2 - \left(1 - \frac{2GM}{c^2 R}\right) dR^2 - R^2 (d\theta^2 + \sin^2\theta \, d\varphi^2) + \frac{4GM}{cR} dR \, dt_E$$

A scenario is then sketched out, should the situation shown in figure 26 become established. The solution would represent a kind of snapshot of a mass inversion and expulsion process in a T-symmetric sheet.

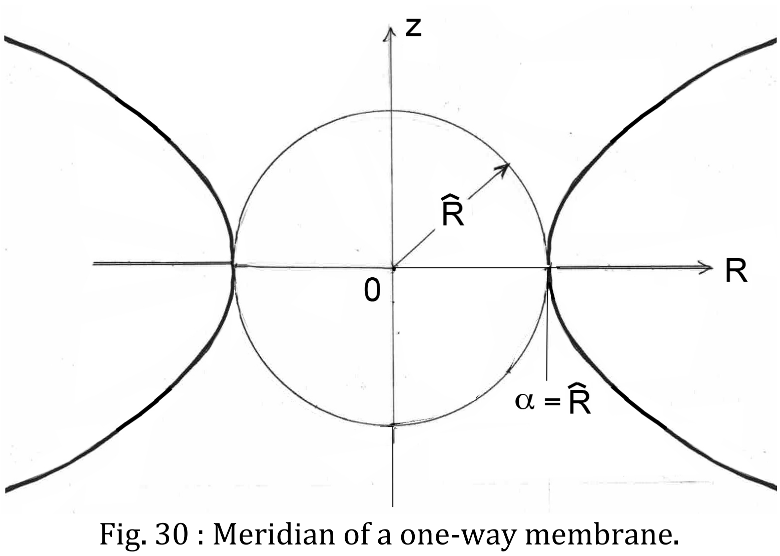


Fig. 30 : Meridian of a one-way membrane.

8 – **Back to the Schwarzschild interior metric** [8].

As it was only available in English translation in 1999, the aspects that follow are still largely unknown to theorists. This metric is expressed in terms of an angle . χ .

Das Linienelement im Innern der Kugel nimmt, wenn man statt x_1, x_2, x_3 (ix) die Variabeln χ, ϑ, ϕ benutzt, die einfache Gestalt an:

$$ds^2 = \left(\frac{3\cos\chi_a - \cos\chi}{2}\right)^2 dt^2 - \frac{3}{\varkappa\rho_0}\left[d\chi^2 + \sin^2\chi\, d\vartheta^2 + \sin^2\chi \sin^2\vartheta\, d\phi^2\right]. \quad (35)$$

Außerhalb der Kugel bleibt die Form des Linienelements dieselbe, wie beim Massenpunkt:

$$\left.\begin{aligned} ds^2 &= \left(1 - \frac{\alpha}{R}\right) dt^2 - \frac{dR^2}{1-\alpha/R} - R^2 (d\vartheta^2 + \sin^2\vartheta\, d\phi^2) \\ \text{wobei:} \quad R^3 &= r^3 + \rho \end{aligned}\right\} \quad (36)$$

ist. Nur wird ρ nach (33) bestimmt, während für den Massenpunkt $\rho = \alpha^3$ war.

Fig.31 : Schwarzschild 1916 interior metric. [8]

Translation :

When one avails of the variables χ, ϑ, ϕ instead of x_1, x_2, x_3 (ix), the line element in the interior of the sphere takes the simple form:

$$ds^2 = \left(\frac{3cos\chi_a - cos\chi}{2}\right)^2 dt^2 - \frac{3}{\kappa\rho_0}[d\chi^2 + sin^2\chi d\vartheta^2 + sin^2\chi sin^2\vartheta d\phi^2]. \quad (35)$$

Outside the sphere the form of the line element remains the same as in "Mass point":

$$ds^2 = (1-\alpha/R)dt^2 - \frac{dR^2}{1-\alpha/R} - R^2(d\vartheta^2 + \sin^2\vartheta d\phi^2) \quad (36)$$
$$\textit{where} \quad R^3 = r^3 + \rho.$$

Now ρ will be determined by (33), while for the mass point it was $\rho = \alpha^3$.

Fig. 31 bis : Schwarzschild 1916 interior metric

Note in passing (red arrow) Schwarzschild's very discreet mention of a purely imaginary time coordinate (ix), which historians of science would do well to comment on.

Next come the variations in pressure and speed of light within the mass:

4. Die Lichtgeschwindigkeit in unserer Kugel wird:

$$v = \frac{2}{3\cos\chi_a - \cos\chi}, \tag{44}$$

sie wächst also vom Betrag $\frac{1}{\cos\chi_a}$ an der Oberfläche bis zum Betrag $\frac{2}{3\cos\chi_a - 1}$ im Mittelpunkt. Die Druckgröße $\rho_0 + p$ wächst nach (10) und (30) proportional der Lichtgeschwindigkeit.

Im Kugelmittelpunkt ($\chi = 0$) werden Lichtgeschwindigkeit und Druck unendlich, sobald $\cos\chi_a = 1/3$, die Fallgeschwindigkeit gleich $\sqrt{8/9}$ der (natürlich gemessenen) Lichtgeschwindigkeit geworden ist. Es ist damit eine Grenze der Konzentration gegeben, über die hinaus eine Kugel inkompressibler Flüssigkeit nicht existieren kann. Wollte man unsere Gleichungen auf Werte $\cos\chi_a < 1/3$ anwenden, so erhielte man bereits außerhalb des Kugelmittelpunktes Unstetigkeiten.

Fig. 32 : Evolution of the speed of light and pressure. [8]

Translation :

4. La vitesse de la lumière dans notre sphère est:

$$v = \frac{2}{3\cos(\chi_a) - \cos(\chi)} \quad (44)$$

de sorte qu'elle varie à partir de la valeur sur la surface $1/\cos(\chi_a)$ jusqu'à la valeur au centre $2/(3\cos(\chi_a) - 1$ · La variable de pression $\rho_0 + p$ augmente selon (10) et (30) proportionnellement à la

Au centre de la sphère $(\chi = 0)$, la vitesse de la lumière et la pression deviennent infinies dès que $\cos(\chi_a) = \frac{1}{3}$, la vitesse de chute est devenue égale à $\sqrt{\frac{8}{9}}$ de la vitesse de la lumière (mesurée naturellement).

il y a donc une limite de densité au-delà de laquelle une boule de fluide incompressible ne peut exister. Si nous voulions appliquer nos équations aux valeurs $\cos(\chi_a) < \frac{1}{3}$, des discontinuités seraient obtenues en dehors du centre de la sphère.

Fig.32bis : Evolution of the speed of light and pressure.

Within the mass, the force of pressure opposes the force of gravity. This was taken up by Oppenheimer and Volkoff [24], and Tolman [23] in their articles published in 1939. This gave rise to the TOV (Tolman-Oppenheimer-Volkoff) equation. This equation also shows that the pressure at the center of the star soars when the conditions for a new criticality, which we will refer to as physical criticality, arise. The various facets of this physical criticality situation are illustrated in this illustration taken from [25]. Physical criticality occurs at:

(42)
$$R_{cr\,phys} = \sqrt{\frac{8}{9}}\,\hat{R} = \sqrt{\frac{8}{9}}\sqrt{\frac{3c^2}{8\pi G\rho}} = \sqrt{\frac{c^2}{3\pi G\rho}}$$

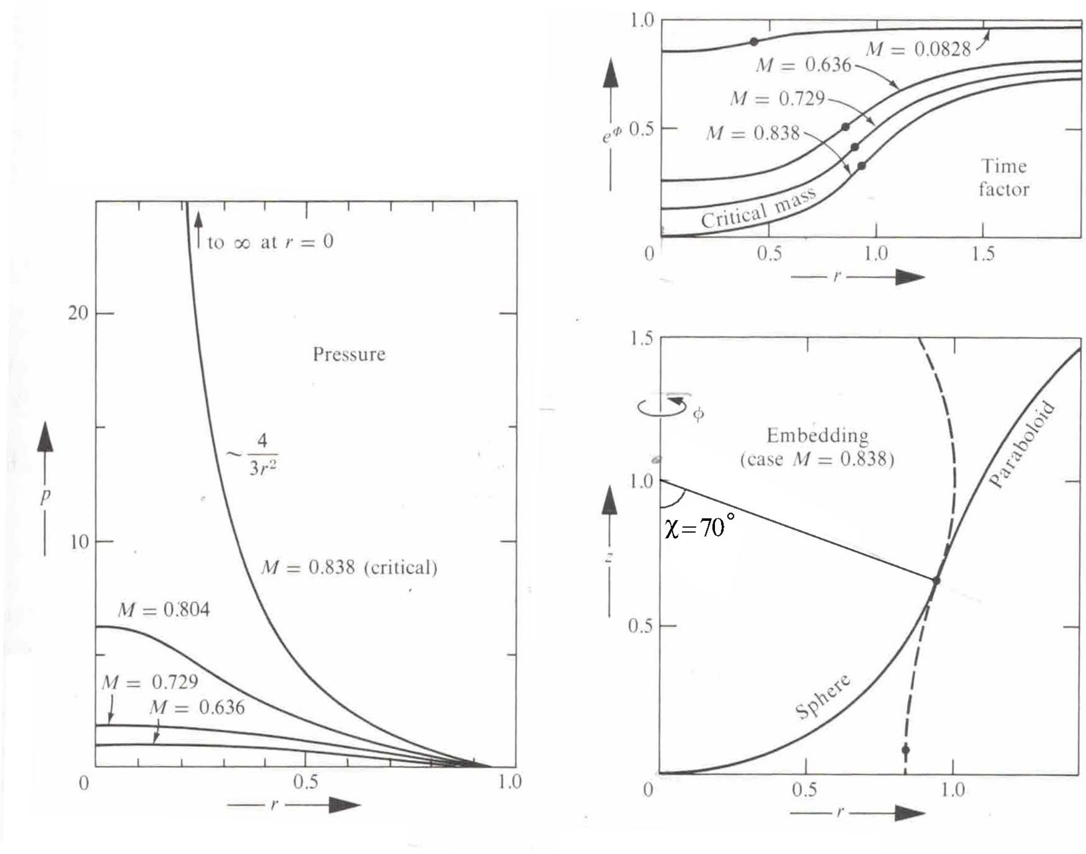


Fig.33 : Physical criticity [25] .

On the left is the pressure surge. Below and to the right is the meridian curve for which physical criticality occurs when χ = arc cos (1/3) which brings this angle value to around 70°, while geometric criticality is reached for $\chi = \pi / 2$. The curve at top right represents the term $\left(3\cos\chi_a - \cos\chi \right) / 2$, such as for an observer at rest :

(43) $$ds = dt\left(3\cos\chi_a - \cos\chi \right) / 2$$

Neutron stars have a density that can roughly be described as almost constant. In his 1916 article, K. Schwarzschild did not hesitate to attribute this rise in pressure to the variation in the speed of light. He is thus the first to envisage the possibility of a variation in this quantity. However, this is considered as an alternative to the inflation model ([26], [27]). In this model, all constants vary jointly, ensuring generalized conservation of all physics equations, as well as conservation of all forms of energy. Assuming that this phenomenon occurs only in the radiative phase, we have the following evolution curves:

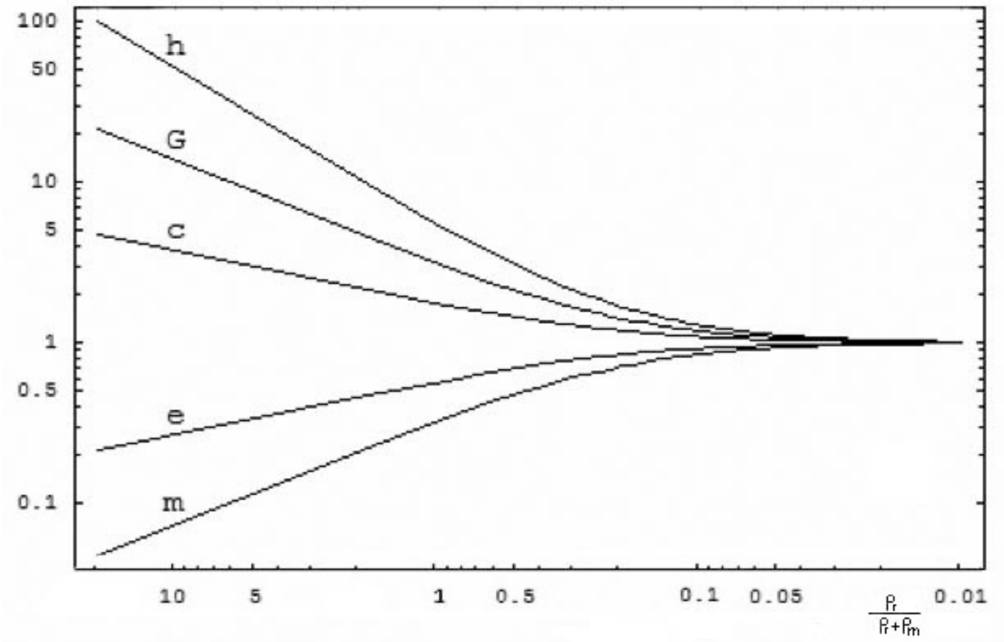


Fig. 34 : Joint normalized changes in the constants of physics
during the radiation dominated era [27].

There's a single observable: the homogeneity of the early universe, linked to the fact that throughout the period preceding decoupling, the cosmological horizon decays as the universe's spatial scaling factor a.

Classically, this surge in pressure is dismissed as unphysical, since the speed of sound, defined as the $(dp / d\rho)^{1/2}$ would then become greater than the speed of light (considered invariant), breaking the principle of causality. In fact, at the center of neutron stars lies a medium where the contribution of the "photon gas" becomes predominant, and where pressure becomes :

(44)
$$p \simeq p_r = \frac{\rho c^2}{3}$$

So, as Schwarzschild suggested, the rise in pressure goes hand in hand with the local variation in the speed of light, allowing the pressure gradient to continue to counterbalance the overwhelming force of gravity... . But what happens when physical criticality is reached and exceeded?

9 – **The plugstar alternative**.

The gravastar model ([33]to [48]), an alternative to the Black Hole model, has aroused great interest in the spécialist community. It eliminates both the cosmological horizon and the central singularity. In consists of a thin layer of conventional matter enclosing a portion of space filled with dark energy, thus preventing implosion. Based on more conventional physical and geometrical considerations, extrapolated from classical design, we present the plugstar model, as another alternative to black holes.

Imagine exceeding the physical criticality by a very small margin. The term in brackets becomes negative. Let's go back to H. Weyl's reasoning: since the length ds (the proper time to within a factor) cannot be reversed, it's the time coordinate that changes sign, meaning that the masses located in this small region become negative. If we consider, as in the Janus model ([30], [31], [32]), that the gravitational field produced by positive masses repels negative counter-masses, interacting with them only by antigravitation, then they will be expelled from the object. The result is a mechanism that ensures the self-stability of objects such as neutron stars, with a very brutal reaction mechanism. Let's take another look at the time factor and imagine overcoming physical criticality.

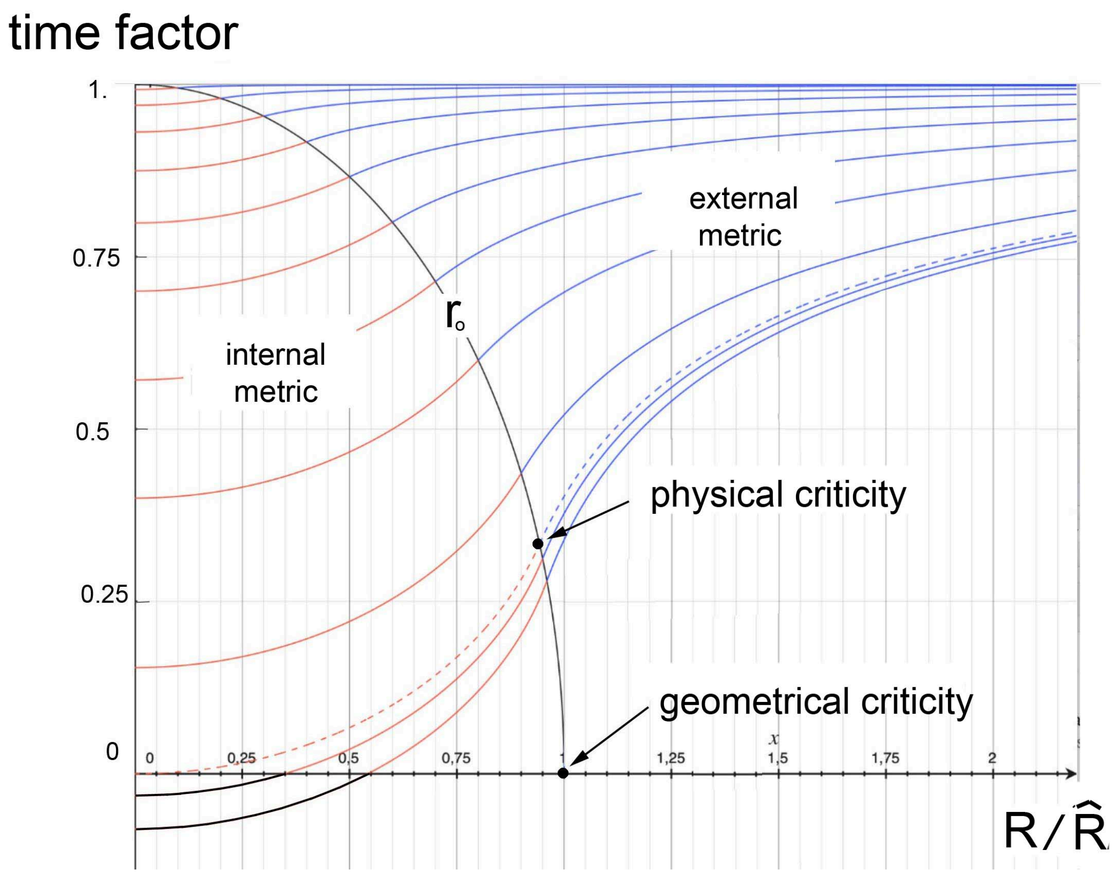


Fig. 35 : Time factor evolution as physical criticality approaches.

The opening mode of this central singularity, where the masses are inverted, is parabolic:

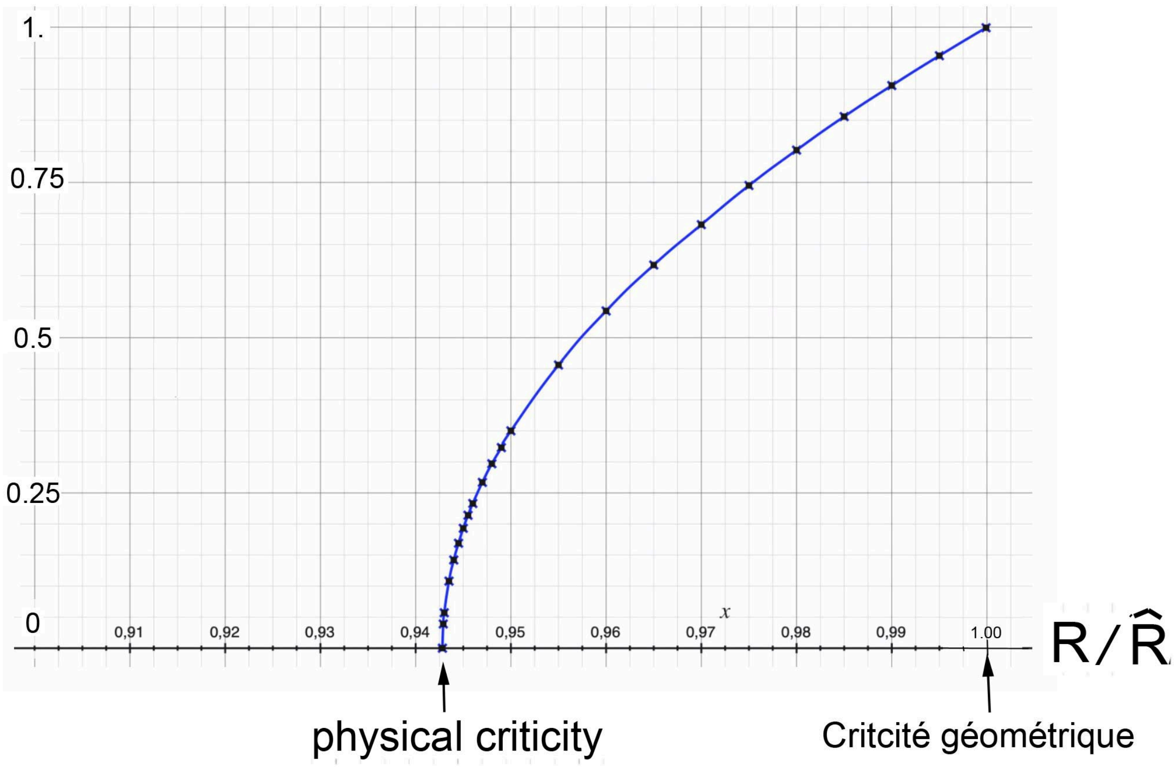


Fig. 36 : Parabolic growth of the diameter of the central singularity.

Physical criticality occurs at:

(45) $$R_{cr\,phys} = \sqrt{\frac{8}{9}}\;\hat{R} = \sqrt{\frac{8}{9}}\sqrt{\frac{3c^2}{8\pi G\rho}} = \sqrt{\frac{c^2}{3\pi G\rho}}$$

All these calculations, concerning the internal metric, correspond to objects with SO(3) symmetry, free from rotation. The values can therefore only be taken as indicative. Neutron stars rotate at speeds of up to a thousand revolutions per second, with peripheral velocities reaching a quarter of the speed of light. By opposing the force of gravity, centrifugal force results in an increase in critical conditions, which is the subject of ongoing research.

10 – **Comparison with observational data**.

The only direct images available are of hypermassive objects at the center of the M 87 and Milky Way galaxies.

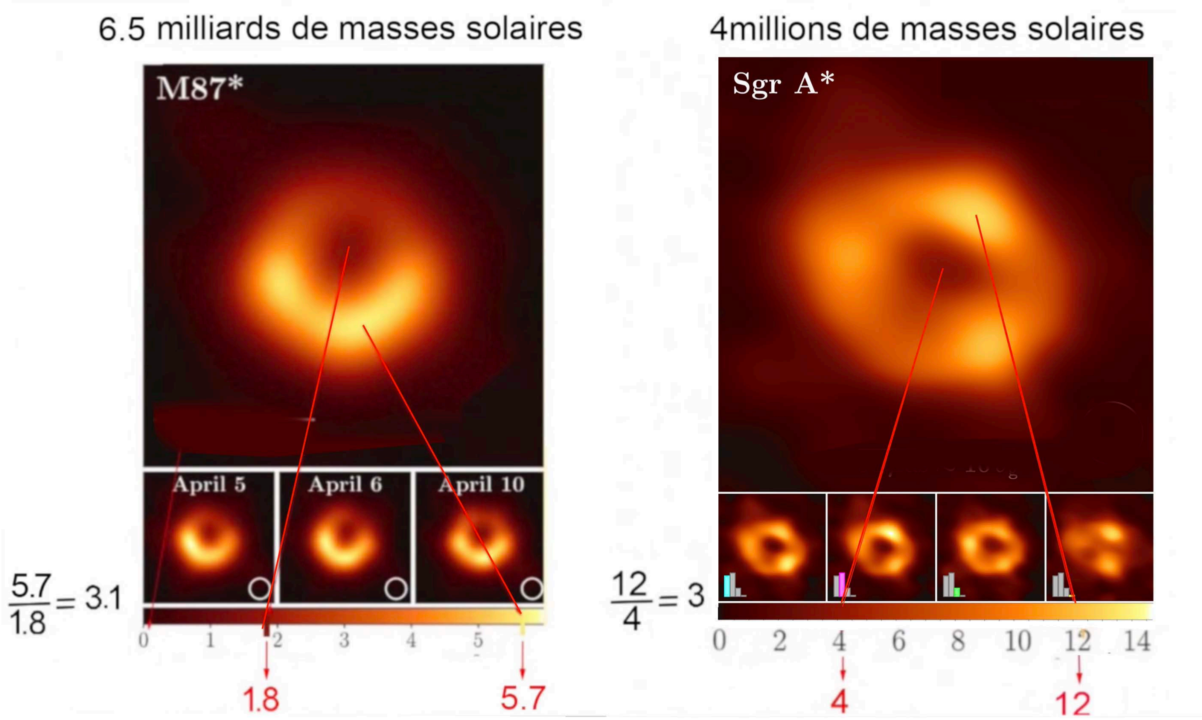


Fig. 37 : Images of objects presented as Giant Black Holes.

In 2019 and 2022, the Event Horizon Telescope system will present for the first time reconstructed images of the two supermassive objects at the center of the M87 galaxy and the Milky Way. What is immediately apparent is that the centers of these objects are not perfectly black. A chromatic scale gives us access, if not to reliable radiation temperature values, at least to the ratios between maximum and minimum values, in both cases. Values close to three are then obtained. It's significant that many articles presenting these results include the words “shadow of giant black hole” in their titles, the argument being that, when they do exist, we can't see what else they might be. As for the light emitted by the central portions, some specialists are quick to attribute it to the gaseous mass in the foreground, corresponding to the accretion disk. But if this is the correct explanation, why does this radiation not betray its presence outside the object?

A quick calculation shows that this darkening of the central part can be interpreted as a gravitational redshift effect emanating from objects at the critical limit.

(46) $$\frac{\lambda'}{\lambda} = \frac{1}{\sqrt{1-\frac{2GM}{R_o c^2}}} \qquad M = \frac{4\pi R_o^{\,3}}{3} \qquad R_o = \sqrt{\frac{c^2}{3\pi G\rho}} \qquad \frac{\lambda'}{\lambda} = \frac{1}{\sqrt{1-\frac{8}{9}}} = 3$$

If this agreement were considered to account for these two observations, it would mean that these objects would be animated by a sufficiently weak rotational motion for this to have little effect on raising the value of the physical critical mass. This raises the question of their origin. The object at the center of galaxy M 87 emits two plasma jets in diametrically opposed directions. It is therefore a quasar. The one at the center of our galaxy is not. Let's think of it as a remnant quasar. In what follows, we are obliged to report on ongoing research, which is still in its early stages. But we had set out the possible scenario decades before. In the Janus model, the cosmos is described using two coupled field equations. This system is likely to generate joint fluctuations $\tilde{g}_{\mu\nu}$ and $\tilde{\tilde{g}}_{\mu\nu}$ in the metrics of the two sectors, resulting in spatio-temporal fluctuations in the way each species influences the other, through its contribution to the gravitational field acting on it. Two possible scenarios can be envisaged, which have already been explored through numerical simulations.

- Either the gravitational field that keeps galaxies together is weakened. In the most extreme cases, these break up completely, producing what are known as irregular galaxies.

- Either the field is strengthened, creating a centripetal density wave. Hoag galaxies could represent this kind of situation.

-

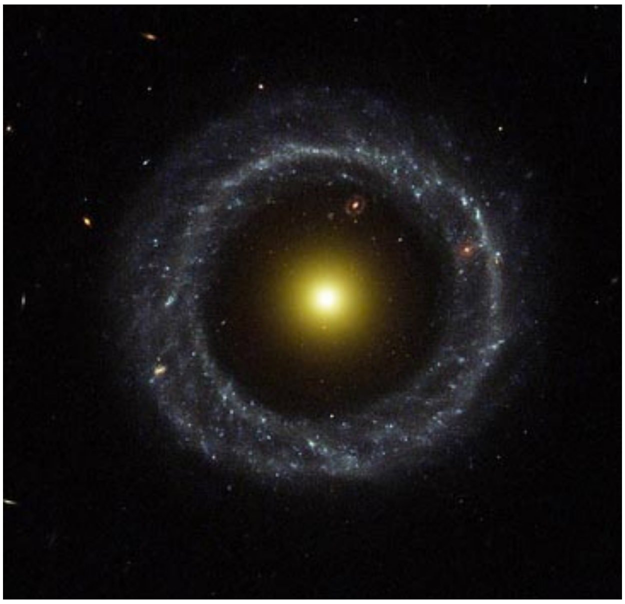


Fig.38 : Hoag galaxy.

The density wave would then converge towards the galactic center. As in spiral structures, the light contrast does not reflect the density contrast, but signals the birth of new stars which, emitting in the ultraviolet, excite the interstellar gas. This emission also ionizes the gas, creating conditions of high magnetic Reynolds number. The density wave thus doubles as an ionization wave, trapping the galaxy's very weak pre-existing magnetic field. The convergence of the density wave gathers the magnetic field lines in the same way as a harvester gathers his ears of wheat. The wave can be compared to a tsunami, which, when it hits the coast, generates considerable effects, but which, when it forms, is only a wave that does not transport matter. As its propagation speed is greater than the residual speed of gas packets, of the order of one km/s, density waves also have the structure of shock waves. This is also the case for spiral density waves, trailings, where the wave front is located in the concave part of the wave. Spiral waves reflect a dynamic friction effect, transferring part of the galaxy's angular momentum to its negative-mass surroundings. But, as shown in the simulations, this effect remains almost imperceptible.

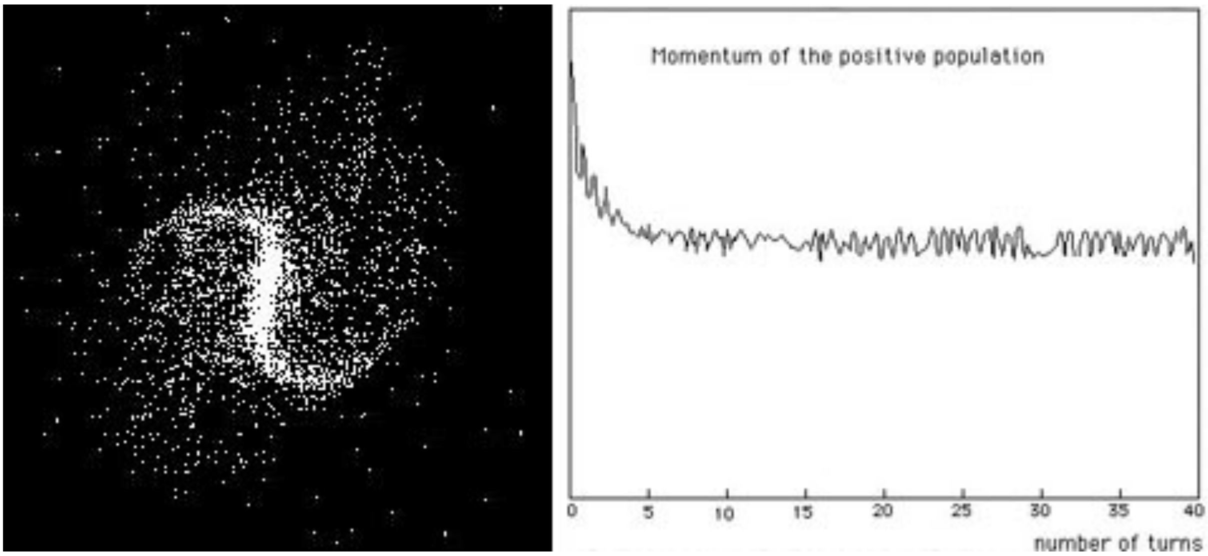


Fig. 39 : Weak loss of angular momentum in galaxies,
due to momentum transfer by density waves [29].

Similarly, centripetal density waves are not very effective at transferring angular velocity from the periphery to the center, as shown by the spiral inhomogeneities visible in Hoag galaxies (see figure 35). When the density wave reaches the center of the galaxy, it results in a sudden rise in density, with fusion reactions starting in a very small volume. The magnetic field, strengthened by the conservation of flux, channels the emitted plasma into two diametrically opposed jets. This is the very essence of the quasar phenomenon. If the concentration of matter is such that physical criticality occurs, the plugstar phenomenon rids the galaxy of excess matter, accompanied by the emission of a powerful gravitational wave. When the quasar phenomenon ceases, a considerable mass remains in its place, whose structure has nothing in common with that of a neutron star. The pressure at its center remains finite, but very high, and the pressure gradient is sufficient to prevent the object collapsing in on itself. Resulting from a quasi-rotation-free object, for the reasons mentioned above, its geometry is then close to the solutions presented by K.Schwarzschild in 1916, i.e. they are identified with subcritical objects generating a gravitational redshift effect with $\lambda'/\lambda \simeq 3$.

This is confirmed by the first observational data mentioned above. It is indeed statistically unlikely, when we opt for the presence of an accretion disk located in the foreground, that its temperature is in both cases such that $\lambda'/\lambda \simeq 3$. We conjecture that these joint fluctuations in metrics were much more intense in the early universe, but that they continue, giving rise at regular intervals to the revival of the quasar phenomenon. This would be the case for the hypermassive, but not hyperdense, object at the center of the Milky Way. We also conjecture, when images of future hypermassive objects become available, that this ratio of maximum and minimum wavelengths will still be close to 3, in other words that they will be subcritical objects with little rotation.

Since the first detections of gravitational waves were made by the LIGO and VIRGO systems, these very faint signals have been decoded by comparing them with those that would result from the merging of neutron stars, then increasingly massive black holes, whose formation scenario the scientific community is struggling to describe. The results show a gap between these two types of event:

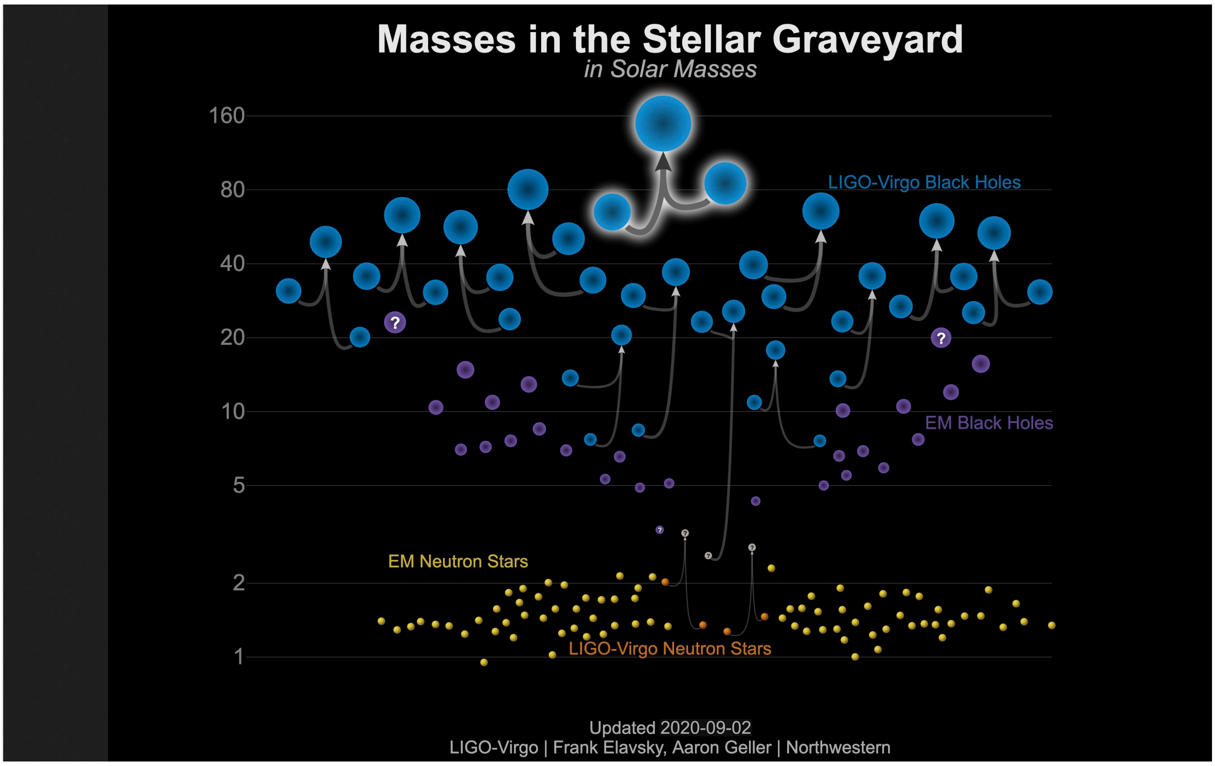


Fig. 40 : Fusion scenarios, after LIGO (2020).

The theory of gravitational waves in the bimetric Janus system as a dissipative phenomenon remains to be constructed, and we are working on it. It will undoubtedly produce a different scenario, in which the most intense gravitational waves are attributed to mass inversions rather than to massive black hole mergers..

11 – **Conclusion**.

This study takes stock of the elements that gave rise to the black hole model, essentially the confusion of the intermediate quantity R in Schwarzschild's original article of January 1916 with a radial coordinate r, which can take on values lower than the Schwarzschild length α. Added to this confusion is the loss of reference points due to the inversion of the signs of the metric's signature and of the common thread represented by the requirement for a real length s, and therefore a real time. Even when it comes to considering the geometry associated with a solution to the Einstein equation with zero second member, considered in isolation, the error has given rise to interpretations betraying a misleading interpretation of the topology of an object, resolutely non-contractile, by envisaging “that inside r may designate the time coordinate and that t may become a space coordinate”.

Finally, the failure to take into account the February 1916 article, and its implications, betraying the impasse of an inescapable event, signalling a physical criticality occurring before the classic geometric criticality could be reached, distorted the construction of any scenario referring either to a subcritical neutron star destabilized by matter input, or to the criticality entry of a massive star at the moment of its collapse. Many articles are currently devoted to so-called gravstars ([33] to[48]) as an alternative to the black hole model. The approach consists in adding an ad hoc negative source term to the field equation, in a spherical portion of space, which would reflect the presence of a local and repulsive concentration of dark energy. . But ther's no explanation of what this is, or how it came out. It is then surrounded by a thin shell of ordinary matter. This has the effect od obliterating the horizon and the central singularity, resulting in an object whose dark central part is no perfectly dark. Revisiting this question gives rise to the alternative model of self-stabilizing plugstars, where any excess matter is eliminated by mass inversion. After justifying the hypothesis of a low incidence of rotation for hypermassive objects located at the center of galaxies, and considering them as subcritical objects, we show that the ratio of maximum and minimum wavelengths, reflecting a gravitational redshift effect λ'/λ , close to three, agrees with available observational data.

Appendix: Comparative diagrams for constructing the zero second member solution of Einstein equation, according to Schwarzschild and Hilbert.

Schwarzschild

Managing geometric assumptions: $\delta\int ds = 0$

$$ds^2 = F^2 dt^2 - G(dx^2 + dy^2 + dz^2) - H(xdx + ydy + zdz)$$

Shift to polar coordinates :

$$x = r\sin\vartheta\cos\phi\,,\ y = r\sin\vartheta\sin\phi\,,\ z = r\cos\vartheta$$

F,G,H fonctions of $r = \sqrt{x^2 + y^2 + z^2}$

$$ds^2 = F dt^2 - (G + H r^2) - G r^2 (d\vartheta^2 + \sin^2\vartheta\, d\phi^2) \qquad (6)$$

$$x_1 = \frac{r^3}{3}\ ,\quad x_2 = -\cos(\vartheta)\ ,\quad x_3 = \phi\ ,\quad x_4 = t \qquad (7)$$

$$ds^2 = f_4 dx_4{}^2 - f_1 dx_1{}^2 - f_2 \frac{dx_2{}^2}{1 - x_2{}^2} - f_3 dx_3{}^2 (1 - x_2{}^2) \qquad (9)$$

$$ds^2 = f_4 dr^2 - f_1 r^4 dr^2 - f_2 d\vartheta^2 - f_3 in^2\vartheta\, d\phi^2$$

$$f_4 = 1 - \frac{\alpha}{(r^3+\alpha^3)^{1/3}} \quad f_2 = \frac{(r^3+\alpha^3)^{-4/3}}{1-(r^3+\alpha^3)^{1/3}} \quad f_2 = f_3 = (r^3+\alpha^3)^{2/3}$$

$$ds^2 = \frac{(r^3+\alpha^3)^{1/3} - \alpha}{(r^3+\alpha^3)^{1/3}} c^2 dt^2 - \frac{r^4}{(r^3+\alpha^3)\left[(r^3+\alpha^3)^{1/3} - \alpha\right]} dr^2$$

Hilbert

Managing the geometrical assumpton and shft to polar coordinates;

$$w_1 = r\cos\vartheta\,,\ w_2 = r\sin\vartheta\cos\phi\,,\ w_3 = r\sin\vartheta\sin\phi\,,\ w_4 = 1$$

$$g_{\mu\nu} = \delta_{\mu\nu} + \varepsilon h_{\mu\nu} \qquad (37)$$

Bilinear form:

$$F dr^2 + G(d\vartheta^2 + \sin^2\vartheta\, d\phi^2) + H dl^2 \qquad (42)$$

F , G , H function of r $\longrightarrow$ $r_* = \sqrt{G(r)}$

$$M(r_*) dr_*{}^2 + r_*{}^2 (d\vartheta^2 + \sin^2\vartheta\, d\phi^2) + W(r_*) dl^2 \qquad (43)$$

$$M = \frac{r_*}{r_* - \alpha} \qquad W = \frac{r_* - \alpha}{r_*}$$

$$G(dr_*, d\vartheta, d\phi, dl) = \frac{r_*}{r_* - \alpha} dr_*{}^2 + r_*{}^2 (d\vartheta^2 + \sin^2\vartheta\, d\phi^2) + \frac{r_* - \alpha}{r_*} dl^2$$

$l = it$ $\downarrow$

$$G(dr_*, d\vartheta, d\phi, dt) = \frac{r_*}{r_* - \alpha} dr_*{}^2 + r_*{}^2 (d\vartheta^2 + \sin^2\vartheta\, d\phi^2) - \frac{r_* - \alpha}{r_*} dt^2$$

$$r^* = (r^3 + \alpha^3)^{1/3}$$

$$\delta\int ds^2 = 0$$

Fig.41 : Compared Schwarzschild and Hilbert schemes.

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