# Positive and negative real, complex, imaginary masses and energies.

J.P.Petit[1] , H.Zejli[2], F.Margnat[34]

---

**Key words:** P-symmetry, T-symmetry, dynamic groups, momentum, coadjoint action, extended Poincaré group, orthochron group, antichron set, Kaluza space, paradigm shift, complex Poincaré group

---

**Abstract :** We review a number of possible avenues for the paradigm shift necessitated by the multiple contradictions that are increasingly weighing down the entire Standard Model, whether in particle physics or cosmology. These are based on the implementation of new symmetries. The convergence between what emanates from quantum mechanics and the theory of dynamical groups, i.e. inversions of mass and energy, is noted. An extension to the complex field is explored.

---

## 1 – Introduction :

On the grave of the great mathematician David Hilbert, in Göttingen, we find his motto, engraved on his tombstone:

*Wir müssen wissen, wir werden wissen*

*We must know and we shall know.*

It's hard to imagine such a limitless ambition. But isn't it true that the knowledge scientists are after is constantly eluding them? To this we can add a phrase attributed to Goethe: "The goal is the path". And this is perhaps what the destiny of the scientist boils down to: trying to make progress in the knowledge - as the history of science attests - that every new step forward will be taken to transcend a new contradiction, while awaiting the next one.

Theoretical physics and cosmology have reached an impasse. For decades, Nature has refused to fit into the mold that theorists have constructed. Particle physics predicted the existence of superparticles. But they refuse to appear. Until now, progress in physics had been linked to an increase in the energy brought into play in particle colliders. Since this failure, what path should we follow? Some at CERN suggest building a collider 91 kilometers in diameter, at a cost of 20 billion euros. The aim? Unspecified. The idea is simply to venture into an unknown field. But what if we don't find anything definite, except even more sprays, reminiscent of the sparks flying from the blacksmith's anvil? In cosmology and astrophysics, dark matter refuses to be captured. Space telescopes have brought a harvest of problems. In 2017, the Hubble telescope revealed the presence of an immense void, a hundred million light-years away, 600 million light-

---

[1] jean-pierre.petit@manaty.net
[2] hicham.zejli@manaty.net
[3] florent.margnat@univ-poitiers.fr
[4] jean-pierre.petit@manaty.net

years from Earth. Subsequent observation confirmed a lacunar universe structure. The James Space Telescope added to the confusion by discovering fully-formed galaxies 300 million years old. And, to round things off, the first images of hypermassive objects at the center of the M87 and Milky Way galaxies, described as giant black holes, don't have perfectly black central parts. So, what could be the way forward? Let's explore a few avenues.

## 2 – **Possible existence of negative-energy states in quantum mechanics**.

In quantum mechanics, negative energy states have long been a subject of controversy. Quantum field theory uses the inversion operators of space (P) and time (T), which can be linear and unitary, or antilinear and antiunitary. Historically, physicists have often avoided negative energy states by arbitrarily choosing linear and unitary P, and antilinear and antiunitary T. However, the discovery of the accelerating cosmic expansion in 2011 has called this approach into question. In the Einstein field equation, this phenomenon can be interpreted by the presence of negative pressure, a form of bulk energy density. In solving the Dirac equation, Paul Dirac discovered solutions with both positive and negative energy terms:

(1) $$E = \pm \sqrt{p^2c^2 + m^2c^4}$$

The solutions with $E = +\sqrt{p^2c^2 + m^2c^4}$ correspond to the positive energies expected for particles, while those with $E = -\sqrt{p^2c^2 + m^2c^4}$ suggest the existence of negative-energy states. This double solution is inherent in any relativistic theory. In quantum mechanics, negative energy would lead to a negative probability density, which is physically unacceptable since negative probabilities do not exist. Indeed, using the Klein-Gordon equation, we have:

(2) $$j^o = \frac{i}{2m}(\psi^* \partial_t \psi - \psi \partial_t \psi^*)$$

If the Ψ wave function is of the plane wave type, we have:

(3) $$\psi(\vec{r},t) = \mathrm{N}\, e^{-i(Et - \vec{p}.\vec{r})}$$

Substituting this function into $j^o$ expression gives:

(4) $$j^o = \frac{i}{2m}\left[\ \psi^*(-iE\psi) - \psi(iE\psi^*)\right] = \left|N\right|^2 \frac{E}{m}$$

This shows that $j^o$ is proportional to E. Thus, a negative energy E leads to a negative probability density, which is not acceptable in quantum mechanics.

Solutions to the Dirac equation pose a similar challenge. Dirac introduced four solutions: two with positive energy and two with negative energy. To get around this problem, Stueckelberg and Feynman proposed that negative-energy particles be interpreted as traveling backwards in time. This means that, by changing the direction of time, negative energy can be transformed into positive energy. Feynman illustrated this idea with diagrams, where particles moving backwards in time are seen as antiparticles. For example, an electron with negative energy can be interpreted as a positron moving backwards in time.

Dirac, however, criticized this approach, asserting that negative-energy states must be taken into account. Each negative-energy state solution corresponds to an antiparticle with an opposite charge.

CPT (charge, parity and time) symmetry plays an important role in quantum physics. The combined C, P and T transformations maintain the coherence of quantum states. For example, C charge conjugation transforms a particle into its antiparticle by inverting the charge:

(5) $$C\psi = i\gamma^2\psi^*$$

By applying this transformation, Dirac solutions can be related to Feynman-Stueckelberg solutions. The $\gamma^5$ transformation, defined by :

(6) $$\gamma^5 = \gamma^o\gamma^1\gamma^2\gamma^3$$

connects positive and negative energy states by inverting the mass:

(7) $$\psi_1^-(m \to -m) = \gamma^5\psi_1^+$$

This transformation suggests that negative energies are acceptable if associated with negative masses.

Thus, through examination of the Dirac and Feynman-Stueckelberg solutions and the discrete symmetries C, P and T, the unitary application of the T operator and antiunitary application of the P operator to the Dirac equation enables particles of positive energy and mass to be transformed into their negative equivalents, while preserving the norm of the quantum state.

This approach is also consistent with the principles of quantum field theory and the use of Dirac fields, showing that the application of a variable electromagnetic potential to a beam of fermions can transform them into negative-mass antifermions. This process involves coupling the creation and annihilation operators for positive and negative mass states, providing an experimental framework for exploring the fundamental nature of matter. [1],[2],[3].

This transform suggests that negative energies are acceptable if they are associated with negative masses..

Thus, by examining the Dirac and Feyman- Stueckelberg solutions and the discrete symmetries C , P , T , and the unitary application of the T operator and the anti-unitary application of the P operator to the Dirac equation makes it possible to transform particles of positive energy and mass into their equivalents, while preserving the norm of the quantum state.
This approach is also consistent with the principles of quantum field theory and the use of Dirac fields, showing that the application of a variable electromagnetic potential to a beam of fermions can transform them into negative-mass antifermions. This process

involves the coupling of creation and annihilation operators for positive and negative mass states, providing an experimental framework for exploring the fundamental nature of matter. Until 2011, this might have seemed justified. But then a Nobel Prize was awarded to the discoverers of the accelerating cosmic expansion ([4],[5],[6]). Now, in Einstein's field equation, such a phenomenon can be interpreted by the presence of a negative pressure, which is a volume density of energy. The study of negative energy states should therefore be considered. The probability of existence of these negative energy states is $E/m$ , m being the mass. This approach would therefore imply the introduction of negative masses into the cosmological model. This has been the subject of articles ([7],[8]), in which we have endeavoured to stick to observational data.

### 3 – **Matter in dynamic group theory**.

In 1970, the mathematician J.M.Souriau [10] introduced a purely geometric definition of the physical quantities energy, impulse and spin. His starting point was the isometry group of Minkowski space. A first subgroup is that of space-time translations.

(8)
$$C = \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

The second subgroup is the Lorentz group. Let X be a vector of the Minkowski space. Let G be the Gramm matrix:

(9)
$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The length of the vector X is:

(10)
$$X^t G X$$

Let's look for a group that preserves this length, whose element is represented by the matrix L that acts on the vector X according to:

(11)
$$X' = L X$$

The conservation of vector length requires that:

(12)
$$X^t G X = X'^t G X' = (LX)^t G L X = X^t (L^t G L) X$$

Hence the axiomatic definition of the Lorentz group:

(13)
$$L^t G L = G$$

Combining the two subgroups gives us the Poincaré group, the isometry group of the Minkowski space:

(14) $$g = \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

In addition to the action:

(13) $$X' = g X$$

We have the action:

(16) $$g^{-1} m g$$

Let be the Lie algebra of the group:

(17) $$\delta g = \begin{pmatrix} \delta L & \delta C \\ 0 & 0 \end{pmatrix}$$

The Poincaré group is of dimension 10. Its Lie algebra is therefore a vector space of the same dimension. The group can be made to act on its Lie algebra :

(18) $$\delta g' = g^{-1} \delta g \, g$$

This gives rise to what Souriau calls the moment of the group, as a dual of Lie algebra. This will enable us to construct the action of the group on its moment, which will also be of dimension 10. Let's calculate the inverse matrix of g.

(19) $$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} L' & C' \\ 0 & 1 \end{pmatrix} = I_4$$

(20) $$g^{-1} = \begin{pmatrix} L^{-1} & L^{-1}C \\ 0 & 1 \end{pmatrix}$$

We can differentiate around any element of the group. Let's place ourselves in the vicinity of the neutral element by posing :

(21) $$L = I + \varepsilon G \omega$$

On a :

(22) $$(I + \varepsilon G \omega)^T G (I + \varepsilon G \omega) = G$$

(23) $$(I + {}^T\varepsilon \omega G) G (I + \varepsilon G \omega) = G$$

(24) $$\omega^T + \omega + \varepsilon \omega^T G \omega = 0$$

To the nearest second order :

(25) $$\omega^T + \omega = 0$$

$\omega$ is an antisymmetric matrix. By posing $\delta C = \gamma$ the element of the Lie algebra becomes

(26) $$\delta g = Z = \begin{pmatrix} G\omega & \gamma \\ 0 & 0 \end{pmatrix}$$

The action on this Lie algebra become:

(20) $$\begin{pmatrix} G\omega' & \gamma' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L^{-1} & L^{-1}C \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} G\omega & \gamma \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

Let :

(27) $$\begin{pmatrix} G\omega' & \gamma' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L^{-1}G\,\omega\,L & L^{-1}G\,\omega C + L^{-1}\gamma \\ 0 & 0 \end{pmatrix}$$

Let's calculate $L^{-1}$ .

(28) $$G\,G = I$$

(29) $$L^{-1}L = I = G\,G = G\,L^{t}G\,L$$

Whence :

(30) $$L^{-1} = G\,L^{t}G$$

(31) $$G\omega' = L^{-1}G\,\omega\,L = G\,L^{t}G\,G\,\omega\,L = G\,L^{t}\,\omega\,L$$

Whence :

(32) $$\omega' = L^{t}\,\omega\,L$$

(33) $$\gamma' = G\,L^{t}G\,G\,\omega C + L^{-1}\gamma = G\,L^{t}G\,G\,\omega C + G\,L^{t}G\,\gamma$$

(34) $$\gamma' = G\,L^{t}\,\omega C + G\,L^{t}G\,\gamma$$

The moment of the Poincaré group is a torsor:

(35) $$\mu \equiv \{ M, P \} \quad \text{with} \quad M^{t} = -M \quad P \in E_4$$

It will be defined by the identity:

(36) $$\mu(Z) \equiv \frac{1}{2} T_r(M.\omega) + P^{t}.G\,\gamma$$

The duality is expressed as follows:

(37) $$\frac{1}{2} Tr(M.\omega) + P^{t}.G\,\gamma = \frac{1}{2} T_r(M'.\omega') + P'^{t}.G\,\gamma'$$

Which gives :

(38)

$$\frac{1}{2}\mathrm{Tr}(M.\omega)+P^{t}.G\,\gamma=\frac{1}{2}T_{r}(M'.L^{t}\omega L)+P'^{t}.(G\,G\,L^{t}\omega C+G\,L^{t}G\,\gamma)$$

$$=\frac{1}{2}T_{r}(M'.L^{t}\omega L)+P'^{t}.L^{t}\omega C+P'^{t}.L^{t}G\,\gamma$$

We can immediately identify on $\gamma$, which gives

(39) $$P^{t}=P'^{t}L^{t}\quad\rightarrow\quad P=L\,P'$$

We know that when we have the trace of a product if we perform a circular permutation within it. We can therefore write:

(40) $$T_{r}(M'L^{t}\omega L)=T_{r}(L\,M'L^{t}\omega)$$

Now let's identify on $\omega$

(41) $$\frac{1}{2}\mathrm{Tr}(M\omega)=\frac{1}{2}\mathrm{Tr}(\,L\,M'\,{}^{t}L\omega)+{}^{t}P\,{}^{t}L\omega C$$

The scalar product of the two vectors ${}^{t}P$ and ${}^{t}L\omega\,C$ is equal to the trace of the matrix formed by treating the vectors in reverse order, i.e.

(42) $${}^{t}P\,{}^{t}L\omega C=\mathrm{Tr}(\,{}^{t}L\omega\;C\;{}^{t}P)$$

In this trace we can perform a circular permutation :

(43) $${}^{t}P\,{}^{t}L\omega C=\mathrm{Tr}(\,C\;{}^{t}P\,{}^{t}L\omega\,)$$

Whence :

(44) $$\frac{1}{2}\mathrm{Tr}(M\omega)=\frac{1}{2}\mathrm{Tr}(\,L\,M'\,{}^{t}L\omega)+\mathrm{Tr}(\,C\;{}^{t}P\,{}^{t}L\omega\,)$$

And :

(45) $$M=L\,M'\,{}^{t}L+2C\;{}^{t}P'\,{}^{t}L$$

We know that any real matrix can be put into the form of a half-sum of a symmetric matrix and an antisymmetric matrix, which gives us:

(46)

$$\frac{1}{2}\mathrm{Tr}(M\omega)=\frac{1}{2}\mathrm{Tr}(\,L\,M'\,{}^{t}L\omega)$$

$$+\frac{1}{2}\mathrm{Tr}\left[(C\,PT\,{}^{t}L+L\,P\,{}^{t}C)\times\omega+(C\;{}^{t}P\;{}^{t}L-L\,P\,{}^{t}C)\times\omega\right]$$

The trace of the product of a symmetrical matrix and an antisymmetrical matrix $\omega$ is zero. This gives:

(47)

$$\mathrm{Tr}\left[(C\,P T\,{}^{t}L + L\,P\,{}^{t}C)\times\omega\;\right] = 0$$

We get :

(48) $$\frac{1}{2}\mathrm{Tr}(M\omega) = \frac{1}{2}\mathrm{Tr}(\,L M'\,{}^{t}L\omega) + \frac{1}{2}\mathrm{Tr}\left[(C\,{}^{t}P\,{}^{t}L - L P^{t}\,C)\times\omega\right]$$

To sum up :

(49) $$M = L M'\,{}^{t}L + C\,P'\,{}^{t}L - L P'\,{}^{t}C$$

(50) $$P = L\,P'$$

We thus construct the action of the Poincaré group on its space of moments. If we wanted to go from P' to P, this would mean using the inverse of the group element. To simplify, we can simply write:

(51) $$M' = L M\,{}^{t}L + C\,P\,{}^{t}L - L P\,{}^{t}C$$

(52) $$P' = L P$$

What is the physical meaning of this dynamic group's action? Spatiotemporal translation places us elsewhere and in a different era. The Lorentz group can be considered as the group of four-dimensional rotations (to which symmetries are added). It conserves the length of the vector P, which will simply be the energy-impulse quadrivector (with c = 1):

(53) $$P = \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

This conservation will result in:

(54) $$E^2 - p^2 = Cst$$

The moment had ten components. One for energy, three for impulse. That leaves six, which we're going to arrange into an antisymmetrical matrix:

(55) $$M = \begin{pmatrix} 0 & -s_z & s_y & f_x \\ s_z & 0 & -s_x & f_y \\ -s_y & s_x & 0 & f_z \\ -f_x & -f_y & -f_z & 0 \end{pmatrix}$$

We then show that the 3-vector f, called "passage" by Souriau, can be cancelled out when we accompany the particle in its motion. It therefore does not characterize a motion, by identifying the nature of a particle with the type of motion considered. The final 3-vector is spin, which is unquantified and emerges as a purely geometric quantity..

(56) $$s = \begin{pmatrix} s_x \\ s_y \\ s_z \end{pmatrix}$$

In that article, we were in pursuit of reality. Here we have Souriau's illustration of the strategy [10], which can be summed up as follows:

- We give ourselves a space, with its metric and isometry group.

- We write the coadjoint action of this group on the dual of its Lie algebra, of the same dimension as this group.

- The components of this moment then define "a physics", one whose "playground" is this particular space.

To illustrate this technique, we'll start with 3D Euclidean space. Its metric is:

(57) $$ds^2 = dx^2 + dy^2 + dz^2$$

Its Gramm matrix is then the unit matrix $I_4$. Its isometry group is the Euclidean group:

(58) $$\begin{pmatrix} a & C \\ 0 & 1 \end{pmatrix}$$

The a matrices are then the orthogonal matrices, defined by:

(59) $$a^t a = I$$

C is the 3D translation vector:

(60) $$C = \begin{pmatrix} \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

Euclid's group, of dimension 6, can be considered as a particular dynamic group, that of statics. The previous calculation can be repeated. The moment is then composed of two 3-vectors, which are then assimilated to a force and a torsional moment. The components of the moment allow summations to be made. In statics, we sum forces and torsional moments; in dynamics, energies, impulses and spins. In 3D, what "inhabits" Euclidean space, considered as related to statics, are states of stress of a uniform, infinite material. In 4D, these are classes of motion, each class being assimilated to a type of particle. These movements are inscribed along geodesics of Minkowski space. Non-zero-length particles are masses, and zero-length particles are photons.

Euclid's group has two related components. A property inherited from its subgroup, that of orthogonal matrices, depending on whether the space is inverted or not, depending on whether we are considering O(3) or SO(3) groups.

Let's return to the Poincaré group.

4 – **Dynamic groups and reality.**

Hilbert saw mathematics as a kind of lamp illuminating the path to knowledge. In his resolutely optimistic attitude, he was convinced that, faced with any problem, a mathematical tool would always emerge that would enable solutions to be found, even if this meant acting as a key to open a door onto a new realm of reality. For example, when dynamic group theory is used to integrate physical quantities into a geometric context, it suggests an extension of the "playground" with new objects. The Lorentz group has four related components.

- The elements $\{ L_n \}$ constitute the neutral component, because, containing the neutral element, they are a subgroup of $\{ L \}$. They invert neither time nor space.

- Elements $\{ L_s \}$ invert space, but not time.

- - Elements $\{ L_t \}$ invert time, but not space.

- Elements $\{ L_{st} \}$ invert both time and space.

By grouping the first two elements, we form :

- The orthochronous group $\{ L_o \} = \{ L_n \} \cup \{ L_s \}$, a subgroup of $\{ L \}$, also known as the restricted Lorentz group.

- The antichronous subset $\{ L_a \} = \{ L_t \} \cup \{ L_{st} \}$

The Poincaré group inherits these properties with four related components. The orthochronous Poincaré subgroup, also known as the restricted Poincaré group, was the only one used by scientists. But what to do with its antichronous components?

We can play with the property:

(61) $$\{ L_t \} = -\{ L_s \} \qquad \{ L_{st} \} = -\{ L_n \}$$

Whence :

(62) $$\{ L_{st} \} = -\{ L_o \}$$

Hence the full Lorentz group representation:

(63) $$\{ L \} = \{ \lambda L_o \} \quad \text{avec} \quad \lambda = \pm 1$$

This leads us to rewrite the complete, or "extended", Poincaré group.» :

(64) $$\begin{pmatrix} \lambda L_o & C \\ 0 & 1 \end{pmatrix}$$

And the action :

(65) $$M' = L_o M \, {}^tL_o + \lambda C \, P \, {}^tL_o - \lambda L_o P \, {}^tC$$

(66) $$P' = \lambda L_o P$$

We thus obtain the physical meaning of the inversion of the time coordinate, of T-symmetry $(\lambda = -1)$ :

- It reverses energy (and therefore mass, through E = m $c^2$) and impulse.

- It leaves the spin unchanged.

The same "message" appears as in quantum mechanics, in the form of an invitation to consider an extension of the physical field to include negative energies and masses.

## 5 – **A further extension of dynamic group theory**.

For the moment, our "real world" is populated only by photons and electrically uncharged masses. As early as 1915, D.Hilbert had attempted to include electromagnetic phenomena in a geometric formalism. As quantum mechanics had not yet emerged, nor had high-energy physics, scientists of the time knew only two forces: the gravitational force and the electromagnetic force. In 1915 and 1916, Hilbert published an essay entitled "Fundamentals of Physics". In the first version, presented on November 20, 1915 [25], was the first presentation of the field equation, which Einstein presented five days later [26], in the same journal. By mutual agreement, Einstein was given the credit for what was to become the foundation of general relativity. Here is Hilbert's published field equation:

Unter Verwendung der vorhin eingeführten Bezeichnungsweise für die Variationsableitungen bezüglich der $g^{\mu\nu}$ erhalten die Gravitationsgleichungen wegen (20) die Gestalt

$$[\sqrt{g}\,K]_{\mu\nu} + \frac{\partial\sqrt{g}\,L}{\partial g^{\mu\nu}} = 0. \tag{21}$$

Das erste Glied linker Hand wird

$$[\sqrt{g}\,K]_{\mu\nu} = \sqrt{g}\,(K_{\mu\nu} - \tfrac{1}{2}\,K g_{\mu\nu}),$$

Fig.1 : David Hilbert's field equation, November 20, 1915 [h]

system in der gewohnten Weise, so erhalten wir an Stelle von (2a) die äquivalenten Gleichungen

$$R_{im} = \sum_l \frac{\partial \Gamma^l_{im}}{\partial x_l} + \sum_{\varrho l} \Gamma^l_{i\varrho}\Gamma^\varrho_{ml} = -\varkappa\left(T_{im} - \frac{1}{2}\,g_{im}\,T\right) \qquad (6)$$

Fig.2 : The same, published by Einstein on November 25, 1915 [i]

Hilbert designates the Ricci tensor and scalar as $K_{\mu\nu}$ and $K$. In Einstein's equation, Einstein places the Laue scalar $T$ in the second member. In Einstein's equation, Einstein places the Laue scalar $T$ in the second member. But Einstein's can be considered the most accomplished. Right up to his death in 1955, Einstein pursued the project of integrating gravitation and electromagnetism into a single field equation, without succeeding. In fact, it's impossible with only 4 dimensions. This extension was attempted in 1921 by Theodor Kaluza [26] and in 1926 by Oskar Klein [28]. The complete calculation was given by J.M.Souriau in 1964 [11]. This is a little-known work, published exclusively in French. This work includes the geometric translation of matter-antimatter symmetry, by inversion of the fifth dimension. Let's extend this dimensional context to a 4D space, known as Kaluza space, where the fifth dimension $\zeta$ is space-like, i.e. where the metric becomes

$$ds^2 = -\,d\zeta^2 + dt^2 - dx^2 - dy^2 - dz^2 - d\zeta^2 \tag{67}$$

The Gramm matrix becomes:

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{68}$$

If we look for the isometry group of such a space, it will be the product of the subgroup of pentadimensional translations:

(69)
$$\Gamma = \begin{pmatrix} \Delta\zeta \\ \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

And the subgroup that could be called "extended Lorentz group" $\Lambda$, axiomatically defined by:

(70)
$$\Lambda^t \Gamma \Lambda = \Gamma$$

It is then possible to reconcile the whole calculus of moment, given that the number of dimensions is not explicitly stated. To begin with, we can use Noether's theorem to say that, if there is a one-parameter subgroup of translations along an additional dimension, this must go hand in hand with the conservation of a scalar q, which we will then assimilate to the electric charge. This charge will constitute an eleventh component of the moment of a dynamical group of dimension 11. But this approach, in all its generality, would break the constancy of the electric charge. We can then consider only one subgroup of this extended Lorentz group:

(71)
$$\begin{pmatrix} \mu & 0 \\ 0 & \lambda L_o \end{pmatrix} \qquad \begin{matrix} \lambda = \pm 1 \\ \mu = \pm 1 \end{matrix}$$

This subgroup acts on the additional scalar, on this electric charge q, in only two ways.

- It preserves this electric charge q

- It inverts it

We can then construct a subgroup of this extension of the complete Poincaré group by combining it with the subgroup of pentadimensional translations.:

(72)
$$\begin{pmatrix} \phi \\ C \end{pmatrix} \qquad C = \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}$$

and denoting by $\phi$ the increment $\Delta\zeta$ :

(73)
$$\begin{pmatrix} \mu & 0 & \phi \\ 0 & \lambda L_o & C \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \lambda = \pm 1 \\ \mu = \pm 1 \end{array}$$

This is a dynamic group where the addition of load reversal symmetry brings the number of related components to 8. Calculating the action on the moment is straightforward. Assuming $\delta\phi = \beta$ the element of the group's Lie algebra is the format matrix (5,5)

(74)
$$Z = \begin{pmatrix} 0 & 0 & \beta \\ 0 & G\omega & \gamma \\ 0 & 0 & 0 \end{pmatrix}$$

The inverse of the group matrix is:

(75)
$$g^{-1} = \begin{pmatrix} -\mu & 0 & -\phi \\ 0 & L_o^{-1} & L_o^{-1}C \\ 0 & 0 & 1 \end{pmatrix}$$

Duality is ensured by the constancy of the scalar:

(76)
$$\frac{1}{2}\mathrm{Tr}\,(M\,\omega) + {}^{t}PG\,\gamma + q\,\varepsilon = \frac{1}{2}\mathrm{Tr}\,(M'\,\omega') + {}^{t}P'G\,\gamma' + q'\varepsilon'$$

The expression of the action of the group on the dual of its lie algebra is finally translated by the addition of an extra dimension:

(77)
$$q' = \mu q$$

(78)
$$M' = L_o M\,{}^{t}L_o + \lambda C\,P\,{}^{t}L_o - \lambda L_o P\,{}^{t}C$$

(79)
$$P' = \lambda L_o P$$

By moving on to Kaluza space and introducing a symmetry on the fifth dimension, we give substance to J.M.Souriau's idea [11]. In Chapter V of [10], Souriau shows how the compactification of this fifth dimension allows us to recover the relativistic Klein-Gordon equation. We can see how dynamical group theory and field theory, including the quantum mechanical version, respond to each other..

## 6 - **The question of paradigm shift**.

As mentioned in the introduction, the failure to detect dark matter, identify dark energy and model the lacunar structure of the universe and the abnormally early birth of stars and galaxies within it, has given rise to the idea of a necessary paradigm shift. This will necessarily involve a revision of the geometric context, since the general relativity model can no longer account for observations. There are two ways of doing this. The first consists in hypothesizing the existence of a second universe, CPT symmetrical to our own, which the authors then situate on the other side of the Big Bang [16]. More recently, N. Kumar [18] envisaged the interaction of these two universes by "entanglement", and showed that this would lead to an acceleration of expansion in our own universe. T-symmetry evokes what had already been predicted by Andréi Sakharov in the 1960s ([19], [20], [21]). The didactic 2D image is as follows:

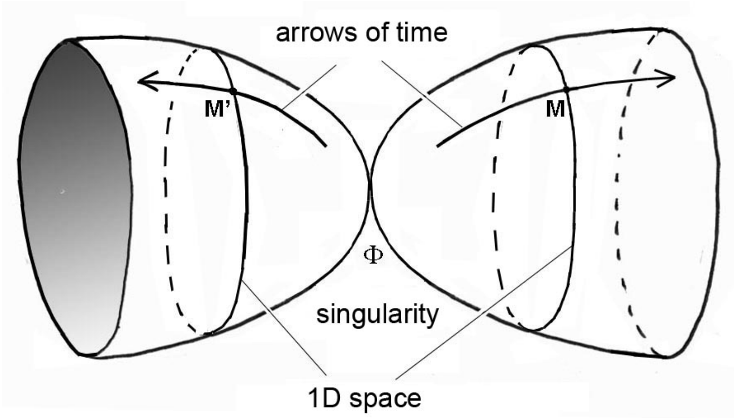


Fig.3 : Didactic 2D image of the Sakharov, Boyle, Finn, Turok and Kumar Model.

Another way of looking at things is the Janus cosmological model ([12], [13], [14], [15], [16]) with a two-folds cover structure.:

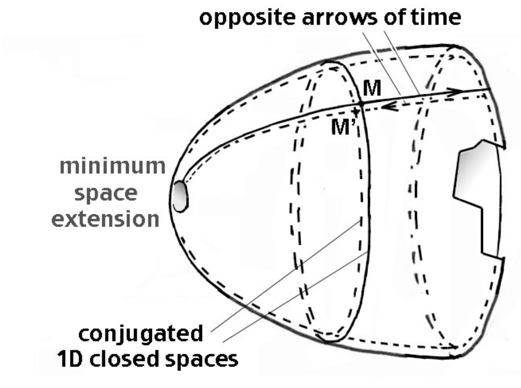


Fig.4 : Didactic 2D image of the Janus Cosmological Model.

Let's translate this CPT-symmetry into dynamic group terms. This gives us the Janus group [7].

(80)
$$\begin{pmatrix} \lambda\mu & 0 & \phi \\ 0 & \lambda L_o & C \\ 0 & 0 & 0 \end{pmatrix} \qquad \begin{array}{l} \lambda = \pm 1 \\ \mu = \pm 1 \end{array}$$

Algebraically, the calculation is very little different. Only the first equation is modified.

(81) $$q' = \lambda \mu q$$

(82) $$M' = L_o M\, {}^tL_o + \lambda C\, P\, {}^tL_o - \lambda L_o P\, {}^tC$$

(83) $$P' = \lambda L_o P$$

CPT-symmetry corresponds to $\lambda = -1$. When compared with observations, this model gives interesting results, including a quantified description of the acceleration of the cosmic expansion [9].

## 7 – **Additional dimensions**.

The electric charge q is just one of the quantum charges, like the baryonic charge $q_b$, the leptonic charge $q_\lambda$ and the muonic charge $q_\mu$. Without the above, we have seen that the appearance of electric charge was the result of the addition of an extra dimension. We can then consider the addition of p extra dimensions, as follows:

(84) $$X = \begin{pmatrix} \zeta^1 \\ \dots \\ \zeta^p \\ t \\ x \\ y \\ z \end{pmatrix}$$

By introducing the vectors "quantum dimension $\zeta$" and "quantum translations $\phi$":

(85) $$\zeta = \begin{pmatrix} \zeta^1 \\ \zeta^2 \\ \dots \\ \zeta^p \end{pmatrix} \qquad \phi = \begin{pmatrix} \phi^1 \\ \phi^2 \\ \dots \\ \phi^p \end{pmatrix}$$

Still restricting the Lorentz group extended to:

(86)

$$\begin{pmatrix} \mu & 0 & 0 & 0 & 0 & \phi_1 \\ 0 & \mu & 0 & 0 & 0 & \phi_1 \\ \dots & \dots & \mu & \dots & \dots & \dots \\ 0 & 0 & 0 & \mu & 0 & \phi_p \\ 0 & 0 & 0 & 0 & \lambda L_o & C \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{matrix} \lambda = \pm 1 \\ \mu = \pm 1 \end{matrix}$$

In passing, we can extend the Janus group into 4 + p dimensions. This translates into the addition of p quantum charges with the relation:

(87)

$$q_i ' = \lambda \mu q_i$$

Moreover, extending Souriau's construction, if these p dimensions are compact, then these p additional loads are quantized.

8 – **Complex extension** [29].

There are numerous works representing attempts to resituate the formalism of general relativity in a complex field, by considering a quantum cosmology ([22],[23],[24]). Let's see what this would look like in the context of dynamical group theory. We'll consider these 4-vectors, belonging to a complex space, with transposed and adjoint notation:

(88)

$$X = \begin{pmatrix} X^\circ \\ X^1 \\ X^2 \\ X^3 \end{pmatrix} \qquad X^T = ( X^\circ, \; X^1, X^2, X^3 ) \qquad \bar{X} = \begin{pmatrix} \bar{X}^\circ \\ \bar{X}^1 \\ \bar{X}^2 \\ \bar{X}^3 \end{pmatrix} \qquad X^* = ( \bar{X}^\circ, \bar{X}^1, \bar{X}^2, \bar{X}^3 )$$

The metric, with a real ds, refers to a Hermite space.:

(89)

$$ds^2 = (dX^\circ)^* dX^\circ - (dX^1)^* dX^1 - (dX^2)^* dX^2 - (dX^3)^* dX^3$$

The real Gramm matrix is therefore:

(90)

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The lenght :

(91) $$< X , X > = X^* G\ X$$

Here again, it is translation-invariant, according to the complex translation vector :

(92) $$C = \begin{pmatrix} \Delta X^\circ \\ \Delta X^1 \\ \Delta X^2 \\ \Delta X^3 \end{pmatrix}$$

Let L be the element L of a group that will change from a vector X to a vector X according to X' = L X . If the length of X is conserved, then this is written :

(93) $$(L\,X)^* G\,L\,X = X^* G\,X$$

which gives the axiomatic definition of a complex Lorentz group:

(94) $$L^* G\,L = G$$

We then construct a complex Poincaré group
 Let L be the element L of a group that will pass from a vector X to a vector X according to X' = L X

If the length of X is conserved, then $< X', Y' > = < X, Y >$ :

(95) $$(L\,X)^* G\,L\,X = X^* G\,X$$

Which gives the axiomatic definition of a complex Lorentz group:

(96) $$L^* G\,L = G$$

We then construct a complex Poincaré group:

(97) $$\begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

The dimension of the complex Lorentz group is 12+4 = 16 (the imaginary components of the main diagonal). The dimension of the complex Poincaré group is 18+8 = 24. The element of its Lie algebra is:

(98) $$Z \equiv \begin{pmatrix} \delta L & \delta C \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \Lambda & \Gamma \\ 0 & 1 \end{pmatrix}$$

(99) $$\delta \left[ L^* G\,L \right] = 0$$

(100) $$(\delta L)^{*} G\, L + L^{*}(G\,\delta L) = 0$$

By differentiating around the unit matrix, it comes to:

(101) $$(\delta L)^{*} G + (G\,\delta L) = 0$$

Or :

(102) $$(G\,\delta L)^{*} + G\,\delta L = 0$$

Therefore $G\,\delta L$ is an anti-Hermitian matrix. The Lie Algebra element is:

(103) $$Z = \begin{pmatrix} G\Omega & \Gamma \\ 0 & 0 \end{pmatrix}$$

$\Omega$ being an anti-hermitian matrix. Let's form the inverse of the group element is : :

(104) $$L^{*} G\, L = G \;\rightarrow\; L^{*} G\, L\, L^{-1} = G\, L^{-1} \;\rightarrow\; L^{*} G = G\, L^{-1} \;\rightarrow\; G\, L^{*} G = G\, G\, L^{-1}$$

(105) $$L^{-1} = \; G\, L^{*} G$$

The coadjoint action of the group on its Lie algebra gives:

(106) $$\begin{pmatrix} G\Omega' & \Gamma' \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} L^{-1} & L^{-1}C \\ 0 & 1 \end{pmatrix} \times \begin{pmatrix} G\Omega & \Gamma \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} L & C \\ 0 & 1 \end{pmatrix}$$

This gives us:

(107) $$\Omega' = L^{*} \Omega\, L$$

(108) $$\Gamma' = G\, L^{*} \Omega C + G\, L^{*} G\, \Gamma$$

In an anti-hermitian matrix, diagonal terms are non-zero.

(109) $$\Omega = \begin{pmatrix} i\omega_{11} & \bar{\Omega}_{12} & \bar{\Omega}_{13} & \bar{\Omega}_{14} \\ -\Omega_{12} & i\omega_{22} & \bar{\Omega}_{23} & \bar{\Omega}_{24} \\ -\Omega_{13} & -\Omega_{23} & i\omega_{33} & \bar{\Omega}_{34} \\ -\Omega_{14} & -\Omega_{24} & -\Omega_{34} & i\omega_{44} \end{pmatrix}$$

So we have these sixteen components, six complex and four pure imaginary.

(110) $$\left\{ \Omega_{sx}\,, \Omega_{sy}\,, \Omega_{sz}\,, \Omega_{fx}\,, \Omega_{fy}\,, \Omega_{fz}, i\omega_{11}, i\omega_{22}, i\omega_{33}, i\omega_{44} \right\}$$

Upper-case letters refer to complex quantities, lower-case letters to real quantities. All in all, we have:

(111) $$Z=\left\{ \Omega_{sx}, \Omega_{sy}, \Omega_{sz}, \Omega_{fx}, \Omega_{fy}, \Omega_{fz}, \Gamma_t, \Gamma_x, \Gamma_y, \Gamma_z, i\omega_{11}, i\omega_{22}, i\omega_{33}, i\omega_{44} \right\}$$

Twenty-four quantities associated with a complex moment :

(112) $$\mu=\left\{ S_x, S_y, S_z, F_x, F_y, F_z, E, P_x, P_y, P_z, i\theta_{11}, i\theta_{22}, i\theta_{33}, i\theta_{44} \right\}$$

Complex energy (and mass):

(113) $$E = e + i\varepsilon \qquad m + i\mu$$

We can form a complex impulse-energy quadrivector:

(114) $$P = \begin{pmatrix} E \\ P_x \\ P_y \\ P_z \end{pmatrix}$$

Combining the vectors S , F , $i\theta$ , we form a complex matrix M

(115) $$M = \begin{pmatrix} i\theta_{xx} & -\overline{S}_z & S_y & F_x \\ S_z & i\theta_{yy} & -\overline{S}_x & F_y \\ -\overline{S}_y & S_x & i\theta_{zz} & F_z \\ -F_x & -F_y & -F_z & i\theta_{tt} \end{pmatrix}$$

Combined with the impulse-energy quadrivector we get the complex moment:

(116) $$\text{complex momentum} \equiv \left\{ M, P \right\} \quad \text{with} \quad M^* = -M \quad ; \quad P \in \mathbb{C}^4$$

We can express duality in terms of a quantity: $\mathcal{M}$

(117) $$\mathcal{M}(Z) = \frac{1}{2}\,\text{Tr}\,(M\,\Omega) + P^*\,G\,\Gamma$$

This translates into:

(118) $$\frac{1}{2}\,\text{Tr}\,(M\,\Omega) + P^*\,G\,\Gamma = \text{Tr}\,(M'\,\Omega') + P'^*\,G\,\Gamma'$$

Whence :

(119) $$\frac{1}{2}\,\text{Tr}\,(M\,\Omega) + P^*\,G\,\Gamma = \frac{1}{2}\,\text{Tr}\,(M'\,L^*\Omega L) + P'^*\,G\,(G\,L^*\Omega C + G\,L^*G\,\Gamma)$$

(120) $$\frac{1}{2}\,\text{Tr}\,(M\,\Omega) + P^*\,G\,\Gamma = \frac{1}{2}\,\text{Tr}\,(M'\,L^*\Omega L) + P'^*\,L^*\Omega C + P'^*\,L^*G\,\Gamma$$

Identification on the $\Gamma$ terms gives the complex relationship:

(121) $$P^* = P'^* L^* \quad \rightarrow \quad P = L P'$$

Or, as justified above:

(115) $$P' = L P$$

(122) $$\|P'\| = P'^* G P' = (LP)^* G (LP) = P^* (L^* G L) P = P^* L P = \|P\|$$

(123) $$\|P\| = (\bar{E}, \bar{P}) G \begin{pmatrix} E \\ P \end{pmatrix} = \bar{E} E - \bar{P} P = (e^2 + \varepsilon^2) - (p^2 + \pi^2) = Cst$$

Let's move on to the M component of the moment. We have:

(124) $$\frac{1}{2} Tr(M \Omega) = \frac{1}{2} Tr(M' L^* \Omega L) + P'^* L^* \Gamma C$$

Like Souriau, we begin by performing a circular permutation in the first term of the second member.:

(125)
$$Tr(M' L^* \Omega L) = Tr(L M' L^* \Omega)$$

Il vient :

(126) $$\frac{1}{2} Tr(M \Omega) = \frac{1}{2} Tr(L M' L^* \Omega) + P^* L^* \Omega C$$

The term $P^* L^* \Omega C$ is the scalar product of two (complex) vectors. There is the row vector $P^*$ and the column vector $L^* \Omega C$. So I can write :

(127) $$P^* L^* \Omega C = Tr(L^* \Omega C P^*)$$

And perform another circular permutation :

(128) $$P^* L^* \Omega C = Tr(C P^* L^* \Omega)$$

Which gives me:

(123) $$\frac{1}{2} Tr(M \Omega) = \frac{1}{2} Tr(L M' L^* \Omega) + Tr(C P^* L^* \Omega)$$

I reverse the ‘ :

(124) $$Tr(M' \Omega) = Tr(L M' L^* \Omega) + 2 Tr(C P^* L^* \Omega)$$

What I can write:

(129) $$Tr\left[(M' - L M' L^* - 2 C P^* L^*) \Omega\right] = 0$$

Or :

(130) $$M' = L M L^{*} + 2C P^{*} L^{*}$$

The physical interpretation of this section remains to be written..

## 9 – **Conclusion**.

What is perhaps most interesting in the overview we have just given is the fact that inversions of energy and mass appear in both quantum field theory and dynamical group theory. An approach which, through the cosmological Janus model, finds its answer in differential geometry. A geometric interpretation of quantum charges and matter-antimatter symmetry is also suggested. Finally, we explore aspects of an extension into the complex field.

## **References**.

[1] B. Guay, "Experimental test to verify whether quantum states of conjugated mass and energy exist," Modern Physics Letters A, vol. 37, no. 03, pp. 2250013, 2022, doi: 10.1142/S0217732322500134, URL: https://doi.org/10.1142/S0217732322500134, eprint: https://doi.org/10.1142/S0217732322500134

[2] N. Debergh, G.D'Agistini, J-P Petit et al., "On evidence for negative energies and masses in the Dirac equation through a unitary time-reversal operator," Journal of Physics Communications, 2018.

[3] N. Debergh, J.P.Petit : On spacetime algebra and its relation with negative mass arXiv :221.07359v1. Reports on Mathematical Physics Vol.91, issue 2, April 2023 pp. 165-181

[4] S. Weinberg, "The Quantum Theory of Fields," vol. 1-3, Foundations, Modern Applications, Supersymmetry, Cambridge University Press, 2000.

[5] Perlmutter, S., et al. 1999, ApJ, 517, 565

[6] Riess, A. G., et al. 1998, ApJ, 116, 1009

[7] B.Schmidt Observational evidence from supernovae for an accelerating universe and cosmological constant. Astron. J. 116, pp 1009-1038 (1998).

[8] J.P.Petit, G.D'Agostini : Negative Mass hypothesis in cosmology and the nature of dark energy. Astrophysics And Space Sccience,. *A* **29**, 145-182 (2014)

[9] G. DAgostini and J.P.Petit : Constraints on Janus Cosmological model from recent observations of supernovae type Ia, Astrophysics and Space Science, (2018), 363:139.https://doi.org/10.1007/s10509-018-3365-3

[10] J.M.Souriau : Structure des systèmes dynamiques. Dunod Ed. France, 1970 and Structure of Dynamical Systems. Boston, Birkhaüser Ed. 1997

[11] J.M.Souriau : Géométrie et Relativité. Ed. Hermann, 1964 (french edition only).

[12] H.Zejli, F.Margnat, J.P.Petit : The Janus Cosmological Model, Mathematically and Physically Consistent. https://hal.science/hal-04583560. Jan. 2024

[13] H.Zejli, F.Margnat, J.P.Petit : The Legacy of Andrei Sakharov. https://hal.science/hal-04572588 may 2024

[14] H.Zejli, F.Margnat, J.P.Petit : Challenging the Standard Model. https://hal.science/hal-04589263 2024

[15] H.Zejli, F.Margnat, J.P.Petit : Plugstars, Alternative to Black Holes. 2024 https://hal-science/hal-04606255

[16] H.Zejli, F.Margnat, J.P.Petit : About the Fundations of Black Holes Theory/ https://hal-science/hal-04637824

[17] L.Boyle, K.Finn, N.Turok. CPT-symmetric universe. Physical Review Letters 121, 251301 20 dec. 2018

[18] N.Kumar On the Accelerated Expansion of the Universe. Gravitation and Cosmology, 2024, Vo.30, n°1 ; pp. 85-88

[19] A.D.Sakharov, ZhETF Pis'ma 5 : 32 ; JETP Lett. 5 : 24 ( 1967 )

[20] A.D.Sakharov , ZhETF Pis'ma 76 : 1172 (1979) ; JETP 49 : 594 (1979)

[21] A.D.Sakharov , (1980). Cosmological Model of the Universe with a Time Vector Inversion. ZhETF (Tr. JETP 52, 349-351) (79): 689–693

[22]C.Jonas, JL. Lehners, J. Quintin. Use of complex metrics ins cosmology. Journal of High Energy Physics. 2022. Vol. 22 n° 284

[23] Edward Witten : A Note on Complex Spacetime Metrics.; arXiv :2111.06514v2 : 11 Feb. 2022

[24] JL.Lehners. Allowable complex metrics in minisuperspace quantum cosmology. Phys. Rev. D , 105, 026022, 25 january 2022.

[25] D.Hilbert. Die Grundlagen der Physik (Esrte Mitteilung). Nachrochten von der Gesellschaft der Wissenschaften zu Göttingen. Mathematisch-ohysikalische Klasse. (1915) 395-407.

[26] A. Einstein : Die Feldgleichungen des Gravitation ( The Field Equation of Gravitation). Sitzungsber. XLIV , 8 , 778 and XLVI s. 799 ( 25 nov 1915)

[27] T.Kaluza. Sitzungsber Preuss Akad. Wiss. Berlin Math. Phys. K1, 966

[28] O.Klein. Z ; Phys. 37, 895

[29] N.Debergh, G.D'Agostini, J-P Petit :On the Poincaré Algebra in a Complex Space-Time Manifold. Journall of Modern Physics 2021,12,pp. 218-228