PT-Symmetry in One-Way Wormholes

Pascal Koiran[1], **Hicham Zejli**[2], **J-P Levy**[3], **Florent Margnat**[4], **M-F Duval**[5], and **Hasnae Zejli**[6]

[1]pascal.koiran@ens-lyon.fr
[2]hicham.zejli@manaty.net
[3]j.p.levy@manaty.net
[4]florent.margnat@univ-poitiers.fr
[5]m-f.duval@manaty.net
[6]hasnae.zejli@manaty.net

June 29, 2024

Keywords : Schwarzschild metric, Einstein's equation, Staticity, Spatiotemporal cross term, Wormhole, White fountain, One-way Membrane, General relativity, Semi-Riemannian spaces, Eddington coordinates, Spacetime geometry, Spacetime topology, Isotropy, Stationarity, Spherical symmetry, T-symmetry, PT-symmetry, Einstein-Rosen Bridge

Abstract

The exterior metric developed by K. Schwarzschild in 1916, as a solution to Einstein's equations in vacuum, is static, having a timelike Killing vector field that is hypersurface orthogonal. This property precludes the presence of a cross term $\mathrm{d}r\,\mathrm{d}t$ in the metric in Schwarzschild coordinates. However, a coordinate transformation by Eddington reintroduces this cross term, revealing that the singularity at the horizon is due to a coordinate choice. In a recent paper, we showed that in Eddington coordinates, the infall time to the horizon is finite from the point of view of a distant observer. In the present paper, we build on this result and on the work of Einstein and Rosen to construct a *Wormhole* and a *White Fountain* as a *One-Way Membrane*. This membrane connects two semi-Riemannian *PT-symmetric* spaces through a *"bridge"* that can be crossed in only one direction.

1 Solutions of Einstein's equation reflecting different topologies

We begin this paper by a review of some the work stemming from the discovery by Schwarzschild of an exact solution to the Einstein field equations in vacuum. The work of Einstein and Rosen [3] is of particular importance for the present paper since we will be interested in the fate of a particle crossing an Einstein-Rosen bridge. At first sight this line of inquiry may look like a dead end to some readers. Indeed, the Einstein-Rosen bridge has often been presented as non-traversable in the literature. In Section 2 we point out that this conclusion is in fact based on an analysis of the Kruskal-Szekeres extension, which as a geometric object is very different from an Einstein-Rosen bridge. The main developments of the paper take place in Sections 3 and 4. We show that a particle crossing the bridge undergoes a PT-symmetry, and we discuss its physical significance. We will in fact not work with the Einstein-Rosen bridge as defined in the seminal paper [3], but with a modified version studied in [9]. One major reason for this modification is that, as explained below in Section 1, the bridge as defined in [3] is not properly glued at the throat in the following sense: as is well known, infalling geodesics do not reach the wormhole throat for any finite value of the Scwharzschild time parameter t. The construction in [9] is inspired by [6, 7] and solves this problem.

In 1916, Karl Schwarzschild successively published two papers ([15],[16]). The first one presented the construction of the solution to Einstein's equation in vacuum, based on the following assumptions:

- *Stationarity*: Independence of the metric terms with respect to the time coordinate, i.e., invariance by time translation.
- *Isotropy* and spherical symmetry, i.e., invariance by *SO(3)*.
- Absence of the $\mathrm{d}r\,\mathrm{d}t$ cross term.
- Lorentzian at infinity.

He rapidly completed this solution, called the exterior Schwarzschild metric, with an interior metric [16] describing the geometry inside a sphere filled with a fluid of constant density ρ_o and a solution to Einstein's equation with a second member. The conditions for connecting the two metrics (Continuity of geodesics) were ensured. The phenomena of the advance of Mercury's perihelion and gravitational lensing confirm this solution. K. Schwarzschild worked to ensure that the conditions governing these two metrics were in accordance with physical reality.

As an example, in the present day, neutron stars, owing to their staggering density and formidable mass, stand as natural cosmic laboratories, probing realms of density and gravity unreachable within terrestrial laboratories. Let us consider two distinct ways through which a neutron star might reach a state of physical criticality.

In a scenario where the star's density, ρ_o, remains constant, a characteristic radius $\hat{r}$ can be defined. Then, a physical criticality is reached when the star's radius is :

$$R_{\mathrm{cr}_\phi} = \sqrt{\frac{8}{9}}\hat{r} = \sqrt{\frac{c^2}{3\pi G\rho_o}} \tag{1}$$

with

$$\hat{r} = \sqrt{\frac{3c^2}{8\pi G\rho_o}} \tag{2}$$

Thus,

- For the exterior metric, it was necessary that the radius of the star be less than $\hat{r}$.
- As for the interior metric, the radius of the star had to be less than R_{cr_ϕ} because a larger radius leads the pressure to rise to infinity at the center of the star.

Next, for massive stars, an imploding iron sphere can present a complex scenario. Assuming the sphere's mass M is conserved during implosion, we must consider two important critical radius :

In the core part, the geometric criticality radius is given by the *Schwarzschild Radius* which is :

$$R_{\mathrm{cr}_\gamma} = R_s = 2\frac{GM}{c^2} \tag{3}$$

Outside of this mass, the physical critical radius is given by 1

With mass conservation expressed as $M = \frac{4}{3}\pi R^3\rho_o$, we can explore how the variable density ρ_o during implosion impacts these critical radius.

Indeed, if physical criticality is reached during implosion, we have $R = R_{\mathrm{cr}_\phi}$. Then, substituting the mass conservation equation into 1, we get :

$$R = R_{\mathrm{cr}_\phi} = 2.25\frac{GM}{c^2} > R_{\mathrm{cr}_\gamma} \tag{4}$$

We can deduce that if the physical criticality is reached for a mass M, then it occurs before geometric criticality appears.

K. Schwarzschild also emphasized that the measurements pertained to conditions far exceeding what was understood within the framework of the astrophysical reality of his time.

It is also important to note that the topology of this geometric solution is built by connecting two bounded manifolds along their common boundary, a sphere S^2 with an area of $4\pi R_o^2$ (*Radius of the star*).

In 1916, Ludwig Flamm considered the external solution as potentially describing a geometric object. The concern was then an attempt to describe masses as a non-contractible region of space ([4]).

In 1934, Richard Tolman was the first to consider a possible handling of the most general metric solution introducing a cross term $\mathrm{d}r\,\mathrm{d}t$. However, for the sake of simplification, he immediately eliminated it using a simple change of variable ([20]).

In 1935, Einstein and Rosen proposed, within the framework of a geometric modeling of particles, a non-contractible geometric structure, through the following coordinate change ([3]):

$$u^2 = r - 2m \tag{5}$$

The metric solution then becomes:

$$\mathrm{d}s^2 = \frac{u^2}{u^2 + 2m}\mathrm{d}t^2 - 4u^2(u^2 + 2m)\mathrm{d}u^2 - (u^2 + 2m)^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{6}$$

The authors thus obtain a non-contractible geometric structure, termed a *"space bridge"*, where a closed surface of area $4\pi\alpha^2$, corresponding to the value $u = 0$, connects two *"sheets"*: one corresponding to the values of u from 0 to $+\infty$ and the other from $-\infty$ to 0. It is noteworthy that this metric is not Lorentzian at infinity[1]. Although this metric, expressed in this new coordinate system, is regular, the authors point out that at the throat surface, its determinant becomes zero. In this geometric structure, two bounded semi-Riemannian sheets are distinguished, the first corresponding to $u > 0$ and the second to $u < 0$. It corresponds to their joining along their common boundary. The overall spacetime does not fit within the standard framework of semi-Riemannian geometry since it does not fulfill the requirement $\det(g_{\mu\nu}) \neq 0$ at the throat. As pointed out in [19], it does fit within the more general framework of singular semi-Riemannian geometry, which allows for degenerate metric tensors.

The Einstein-Rosen bridge (6) satisfies the Einstein field equations in vacuum on both sheets $u > 0$ and $u < 0$. However, there is an issue with the field equations at the throat $u = 0$ since $\det(g_{\mu\nu})$ vanishes there, and this determinant appears in the denominator of the field equations. This issue was already recognized by Einstein and Rosen, and their proposed solution was to work with a form of the field equations that is denominator-free (see equations (3a) in [3],

[1] For this reason, the change of variables $r^2 = \rho^2 + 4m^2$ was proposed by Chruściel ([1], page 77) as an alternative to (5). See also Section 4 of the present paper, where we propose an alternative to the change of variables from [1].

and the paragraph after (5a)). These modified field equations (called nowadays the "polynomial form of the field equations") are satisfied everywhere, including at the throat. It was discovered much later that one can also work with the original form of the field equations if a thin shell of "exotic matter" is added at the throat [6, 7].

As a spacetime, the Einstein-Rosen bridge (6) suffers from the problem that the time coordinate t becomes infinite on the throat (since infall time to the throat is infinite in Schwarzschild coordinates). In (6) the 4-dimensional sheets $u > 0$ and $u < 0$ are therefore not properly glued at the throat.[2] Namely, studying the passage of a particle from the sheet $u > 0$ to $u < 0$ would require to go through $t = \infty$, which is not a well defined part of the manifold. We will see how to fix this problem in Section 3.

In 1939, Oppenheimer and Snyder, capitalizing on the complete decoupling between proper time and the time experienced by a distant observer, in the absence of a cross term in $\mathrm{d}r\,\mathrm{d}t$, suggested using the external metric solution to describe the *"freeze frame"* of the implosion of a massive star at the end of its life. By considering that the variable t is identified with the proper time of a distant observer, it creates this *"freeze frame"* pattern such as a collapse phenomenon whose duration, in proper time, measured in days, seems for a distant observer to unfold in infinite time ([13]). This paper was considered as the foundation of the black hole model.

In 1960, Kruskal extended the geometric solution to encompass a contractible spacetime, organized around a central singularity corresponding to $r = 0$. The geodesics are extended for $r < \alpha$. The black hole model (with spherical symmetry[3]) then takes its definitive form as the implosion of a mass, in a brief moment, perceived as a *"freeze-frame"* by a distant observer ([10]). The Schwarzschild sphere is then termed the *"event horizon"*.

In 1988, M. Morris and K. S. Thorne revisited this geometric interpretation by abandoning contractibility, not to attempt to obtain a geometric modeling of the solution, but to study the possibility of interstellar travel, through *"wormholes"*, using the following metric ([12]):

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}l^2 + (b_o^2 + l^2)(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \tag{7}$$

By focusing their study on the feasibility of interstellar travel, the authors highlight the enormous constraints associated with such geometry as well as its unstable and transient nature.

[2] Gluing the spatial (3-dimensional) parts of the two sheets does not raise any special difficulty, however.

[3] In 1963, Roy Kerr constructed the stationary axisymmetric solution to Einstein's equation in vacuum. However, in this article, we limit ourselves to the interpretations of the stationary solution with spherical symmetry.

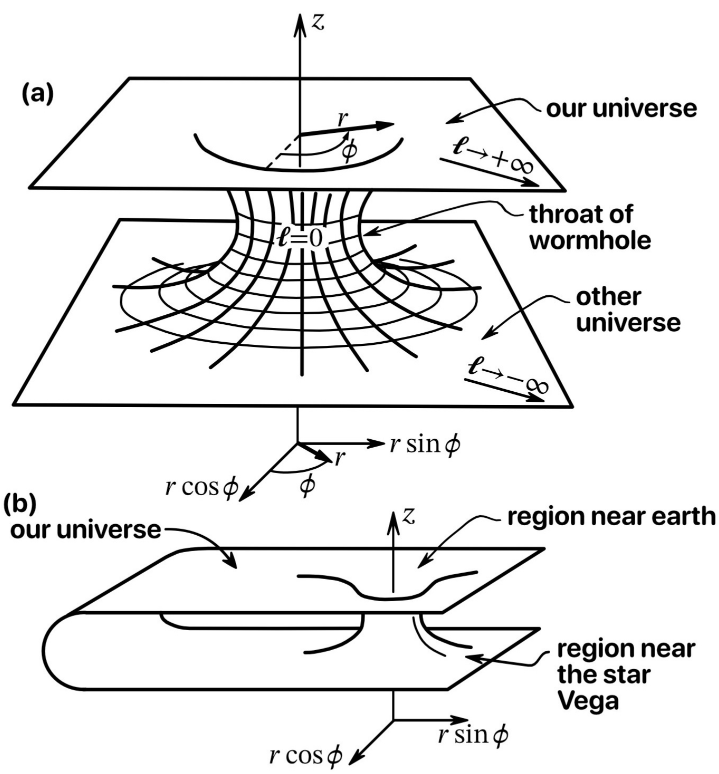

Figure 1: Page 396 of the article by M. Morris and K.S. Thorne (1988)

2 Distinction between the Kruskal-Szekeres Extension and the Einstein-Rosen Bridge

The Kruskal-Szekeres extension and the Einstein-Rosen bridge are two major constructions in the study of spacetime geometry around a wormhole. However, their geometric natures differ significantly.

The Kruskal-Szekeres spacetime is defined by a traditional *semi-Riemannian manifold*, characterized by a non-degenerate metric at every point. This makes it consistent with the general framework of general relativity, where the metric's signature is homogeneous and does not vary [11], [21].

In contrast, the Einstein-Rosen spacetime has a degenerate metric at certain points, namely, at the bridge's throat. This characteristic places it in the class of *singular semi-Riemannian manifolds* as defined by Ovidiu Stoica [19]. This fundamental distinction shows that the Kruskal-Szekeres spacetime is not simply an extension of Einstein-Rosen but a fundamentally different construction.

This geometric difference between the two spacetimes is also responsible for a physical difference. Indeed, as already mentioned in Section 1, for the field equations to be satisfied at the throat of the Einstein-Rosen bridge one needs to add a thin shell of exotic matter at the throat [6, 7] (or one can work with the polynomial form of the field equations as in [3]). By contrast the Kruskal-Szekeres extension satisfies the ordinary form of the field equations in vacuum, including at the event horizon.

Thus, these two spacetimes cannot be considered versions of each other but rather two distinct interpretations of the geometry around a wormhole. This was already pointed out in several papers by Guendelman et al. Consider in particular [6], where they write:

> [29] The nomenclature of "Einstein-Rosen bridge" in several standard textbooks (e.g. [15]) uses the Kruskal-Szekeres manifold. The latter notion of "Einstein-Rosen bridge" is not equivalent to the original construction in [14]. Namely, the two regions in Kruskal-Szekeres space-time corresponding to the outer Schwarzschild space-time region ($r > 2m$) and labeled (I) and (III) in [15] are generally *disconnected* and share only a two-sphere (the angular part) as a common border ($U = 0$, $V = 0$ in Kruskal-Szekeres coordinates), whereas in the original Einstein-Rosen "bridge" construction the boundary between the two identical copies of the outer Schwarzschild space-time region ($r > 2m$) is a three-dimensional hypersurface ($r = 2m$).

We can also cite two other papers whose authors make the same observation regarding the Kruskal-Szekeres extension's inadequacy in properly analyzing the Einstein-Rosen bridges: that of Guendelman et al. [7] and that of Poplawski [14]. Indeed, to distinguish these spacetimes, Poplawski uses the terms *"Schwarzschild bridge"* and *"Einstein-Rosen bridge"*.

For all these reasons, we will *not* work with the Kruskal-Szekeres extension in this paper. We note in particular that the common claim [5, 11] that the Einstein-Rosen bridge is not traversable is actually based on an analysis of the Kruskal-Szekeres extension; but, as pointed out in [7, 9], the original Einstein-Rosen bridge [3] is in fact traversable.

3 Construction of a Lorentzian geometric solution at infinity with two sheets

In this section we study the symmetries of a modified version of the original Einstein-Rosen bridge [3]. Let us consider the exterior Schwarzschild metric in its classical form under the signature *(+ - - -)* :

$$\mathrm{d}s^2 = \left(1 - \frac{\alpha}{r}\right) c^2 \mathrm{d}t^2 - \left(1 - \frac{\alpha}{r}\right)^{-1} \mathrm{d}r^2 - r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2) \tag{8}$$

As recalled in Section 2, Einstein and Rosen defined their "bridge" from the change of variables $r = \alpha + u^2$ in 8. The starting point for the definition of the modified bridge is the idea from [6, 7] to work instead with the change of variables $r = \alpha + |\eta|$ where $\eta \in \mathbb{R}$ is a new radial parameter. As shown in [6, 7], the resulting spacetime satisfies the original form of the field equations, including at the throat $\eta = 0$, if some "exotic matter" (a lightlike membrane) is added at the throat. By contrast, as recalled in Section 2, Einstein and Rosen had to work with the polynomial form of the field equations.

As we now explain, the modified bridge studied in this section is obtained by combining the change of variables $r = \alpha + |\eta|$ with Eddington's change of variables for the time parameter.

3.1 PT-symmetry

Eddington [2] introduced his change of variables

$$t_E^+ = t + \frac{\alpha}{c} \ln \left| \frac{r}{\alpha} - 1 \right| \tag{9}$$

with the aim of eliminating the coordinate singularity at the Schwarzschild surface in $r = \alpha$. The metric becomes:

$$\mathrm{d}s^2 = \left(1 - \frac{\alpha}{r}\right) c^2 \mathrm{d}{t_E^+}^2 - \left(1 + \frac{\alpha}{r}\right) \mathrm{d}r^2 - \frac{2\alpha c}{r} \mathrm{d}r \mathrm{d}t_E^+ - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) \tag{10}$$

We know that under these conditions free fall time is finite in Eddington coordinates, i.e., a massive infalling particle will reach the surface $r = \alpha$ for a finite value of t_E^+ [9]. It is however well-known that the surface $r = \alpha$ is not reached for any finite value of the Schwarzschild time parameter t.

By contrast, escape time for t_E^+ remains infinite. The metric for which the escape time is finite will be obtained by performing this change of variable:

$$t_E^- = t - \frac{\alpha}{c} \ln \left| \frac{r}{\alpha} - 1 \right| \tag{11}$$

In this case, the metric becomes:

$$\mathrm{d}s^2 = \left(1 - \frac{\alpha}{r}\right) c^2 \mathrm{d}{t_E^-}^2 - \left(1 + \frac{\alpha}{r}\right) \mathrm{d}r^2 + \frac{2\alpha c}{r} \mathrm{d}r \mathrm{d}t_E^- - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) \tag{12}$$

The modified bridge studied in [9] combines the change of variables $r = \alpha + |\eta|$ with (9), i.e., we work with the new time parameter $t' = t + \frac{\alpha}{c} \ln \left| \frac{\eta}{\alpha} \right|$.

Thus, the metric becomes:

$$ds^2 = \frac{|\eta|}{\alpha + |\eta|} c^2 dt'^2 - \frac{2\alpha + |\eta|}{\alpha + |\eta|} d\eta^2 - \frac{2\alpha c}{\alpha + |\eta|} d\eta \, dt' - (\alpha + |\eta|)^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right). \tag{13}$$

This line element already appears in the appendix of [6] in slightly different notation. It describes a spacetime made of two sheets connected at the throat $\eta = 0$. The sheet $\eta > 0$ is equipped with the ingoing Eddington metric (10) and the sheet $\eta > 0$ is equipped with the outgoing metric (12). As pointed in [9], an infalling particle beginning its trajectory in the region $\eta > 0$ will reach the throat $\eta = 0$ for a finite value of the Eddington time parameter t', and will then continue in the region $\eta < 0$. This resolves the gluing problem that was mentioned in Section 1 for the original version of the Einstein-Rosen bridge (recall that the throat is reached for $t = \infty$ with the bridge as defined in [3]).

The line element (13) is invariant under the joint transformations $\eta \mapsto -\eta, t' \mapsto -t'$. The physical significance of this symmetry will be discussed in Section 3.3. Note that the line element (6) has a similar symmetry, and in fact it is *more* symmetric since it is invariant under each of the two transformations $u \mapsto -u$ and $t \mapsto -t$. This extra symmetry is due to the absence in (6) of a cross term such as the term $d\eta dt'$ in (13).

3.2 Change of orientation

In general, we expect a P-symmetry or PT-symmetry to be associated to a change of orientation. In this section we confirm that this is indeed the case by taking a closer look at the geometry of the modified bridge (13) in the vicinity of the throat $\eta = 0$. In this representation, the radial geodesics of the first sheet are orthogonal to the tangent plane at the *"space bridge"* when they reach it. These same geodesics, emerging in the second sheet, are also orthogonal to this same tangent plane. Let's now consider four points forming a tetrahedron, which converge towards the *"space bridge"* along radial trajectories. We can set a 3D orientation by defining a direction of traversal of the points on each of the equilateral triangles forming the tetrahedron. With respect to the coordinate $r = \alpha + |\eta|$, it seems as if these points bounce off a rigid surface, leading to an inversion of the orientation of the tetrahedron. The upstream and downstream tetrahedra then become *enantiomorphic* (Figure 2).

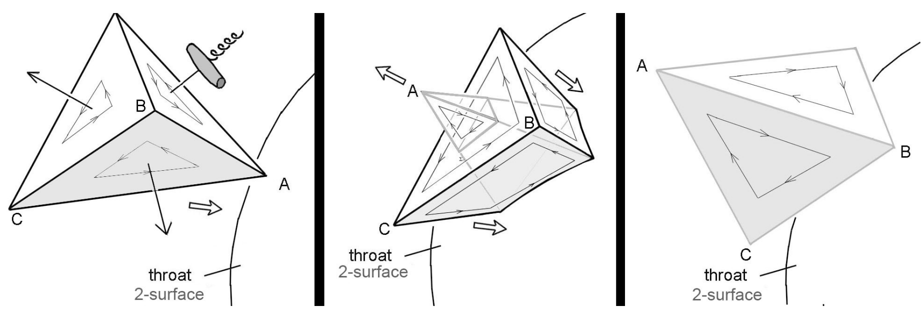

Figure 2: Inversion of space when crossing the *"space bridge"*

The change of orientation is already visible in the simplified 2-dimensional representation of a wormhole in Figure 1. Let us look at this figure from above, and imagine a triangle gliding on the surface of the top sheet toward the throat. After crossing the throat, the triangle starts gliding on the bottom sheet and we now see it upside now from our position above the top sheet. From our point of view, its orientation has therefore changed. The physical meaning of this change of orientation will be discussed in Section 3.3.

The geometric structure of the pair of metrics 6 and 8 thus represents a *"bridge"* connecting two *PT-symmetric* semi-Riemannian spaces.

The element of this 2D-surface is then given by:

$$\sqrt{|\det(g_{\mu\nu})|} = \sqrt{|g_{\theta\theta} g_{\phi\phi}|} = \alpha^2 \sin(\theta) \tag{14}$$

As this metric describes a 2D-surface sphere (like a sphere of constant radius in a 4D spacetime), then the differential area element is given by :

$$\mathrm{d}A = \sqrt{|\det(g_{\mu\nu})|}\mathrm{d}\theta\mathrm{d}\phi = \alpha^2 \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi \tag{15}$$

To find the minimal area of this *"space bridge"*, we must integrate this area element over all possible angles :

$$A = \int_0^{2\pi} \int_0^{\pi} \alpha^2 \sin(\theta)\mathrm{d}\theta\mathrm{d}\phi = 4\pi\alpha^2 \tag{16}$$

It's therefore non-contractible with a minimal area of $4\pi\alpha^2$.

3.3 Identification of the two sheets

In Section 3.2 we have described the change of orientation of a tetrahedron crossing the wormhole throat in Figure 2, and of a triangle crossing the throat in Figure 1. The change of orientation of the triangle is only visible for a person looking at Figure 1 in its entirety. Therefore, it does not correspond to any physically observable phenomenon since any physical observer must be located on one of the two sheets and cannot see directly the other sheet. The situation is the same in Figure 2 : The middle picture represents the situation from a point of view where we could look simultaneously at the two sides of the wormhole (B and C have not reached the throat yet, while A has already crossed it and emerges on the other side). This is again impossible for a physical observer: it seems that the PT-symmetry as described so far does not correspond to any physically observable phenomenon. We can however give it a real physical meaning with an additional ingredient due to Einstein and Rosen [3].

Recall that their motivation was not to investigate interstellar travel as in Figure 1, but to describe elementary particles by solutions to the equations of general relativity. Quoting from the abstract of their paper: *"These solutions involve the mathematical representation of physical space by a space of two identical sheets, a particle being represented by a "bridge" connecting these sheets."*

Einstein and Rosen also suggest that the multi-particle problem might be studied by similar methods, but this work is not carried out in their paper.

Quoting again from [3] : *"If several particles are present, this case corresponds to finding a solution without singularities of the modified Eqs. (3a), the solution representing a space with two congruent sheets connected by several discrete "bridges.""* From their point of view, two points in the mathematical representation (6) with identical values of θ, ϕ but opposite values of u therefore correspond to two points in physical space with the same value of r ($r = u^2 + 2m$). If we make the same identification of points with opposite values of u, the situation represented in the middle picture of Figure 2 can be seen by a physical observer. The change of orientation described in Section 3.2 now has a real physical meaning. We will elaborate on the interpretation of the combined *PT-symmetry* in the next section. In Section 5 we present a precise mathematical model of the identification of the two sheets for the modified bridge described in Section 3.1.

4 Another Representation of this Geometry

By performing the following change of variable to 10 and 12 :

$$r = \alpha\,(1 + \log\,\mathrm{ch}(\rho)) \tag{17}$$

We then obtain the following two metrics:

(18)
$$\mathrm{d}s^2 = \left(\frac{\log\,\cosh(\rho)}{1+\log\,\cosh(\rho)}\right) c^2 \mathrm{d}{t_E^+}^2 - \left(\frac{2+\log\,\cosh(\rho)}{1+\log\,\cosh(\rho)}\right)\alpha^2 \tanh^2(\rho)\mathrm{d}\rho^2 - 2c\alpha\left(\frac{\tanh(\rho)}{1+\log\,\cosh(\rho)}\right)\mathrm{d}\rho\mathrm{d}t_E^+ - \alpha^2(1+\log\,\cosh(\rho))^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)$$

(19)
$$\mathrm{d}s^2 = \left(\frac{\log\,\cosh(\rho)}{1+\log\,\cosh(\rho)}\right) c^2 \mathrm{d}{t_E^-}^2 - \left(\frac{2+\log\,\cosh(\rho)}{1+\log\,\cosh(\rho)}\right)\alpha^2 \tanh^2(\rho)\mathrm{d}\rho^2 + 2c\alpha\left(\frac{\tanh(\rho)}{1+\log\,\cosh(\rho)}\right)\mathrm{d}\rho\mathrm{d}t_E^- - \alpha^2(1+\log\,\cosh(\rho))^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\varphi^2)$$

Therefore, to obtain the metric that structures the second sheet for $\rho < 0$ in order to ensure the continuity of the geodesics translating the transit of matter through the *"bridge"* with a finite escape time on this sheet, we must apply the *T-symmetry* where the time coordinate is reversed upon crossing, i.e., $t_E^+ = -t_E^-$.

This *T-symmetry* is not merely a result of the coordinate choice but has a deeper physical implication. It indicates that the two sheets are time-reversed versions of each other, meaning that events in one sheet are mirrored as reversed in time in the other sheet. This is crucial for maintaining the overall causal structure and physical consistency of the spacetime. The time reversal symmetry ensures that particles traversing the bridge experience a continuous and consistent physical law, even though their time direction is inverted upon crossing.

In the context of our construction, this *T-symmetry* implies that an observer in one sheet would perceive the events in the other sheet as happening in reverse temporal order. This is more than a mathematical artifact; it reflects a physical reality where the symmetry of time reversal plays a fundamental role in connecting the two sheets through the bridge.

Jean-Marie Souriau's dynamic group theory provides further insight into this phenomenon, showing that the inversion of the time coordinate results in an inversion of energy, transforming the motion of a particle of mass m into a motion of a particle of mass $-m$ [18]. This suggests that the *T-symmetry* might lead to observable effects where particles and antiparticles are connected through the bridge.

Those metrics, which are Lorentzian at infinity, structures two sheets corresponding to ρ varying respectively from 0 to $+\infty$ and $-\infty$ to 0. On the *"space bridge"* for $\rho = 0$, the components g_{tt} and $g_{\rho\rho}$ of the metric tensor disappear, leaving only the last two spatial components $g_{\theta\theta}$ and $g_{\phi\phi}$, which are:

$$g_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -\alpha^2 & 0 \\ 0 & 0 & 0 & -\alpha^2 \sin^2\theta \end{pmatrix} \tag{20}$$

On this particular coordinate system, we can infer that its determinant is zero. The *P-symmetry* arises from the fact that adjacent points, this time explicitly differentiated, are inferred by $\rho \to -\rho$. This transformation plays the same role as $u \to -u$ in 6.

One nice property of the change of variables (17) is that the two resulting metrics are explicitly Lorentzian as $|\rho| \to +\infty$. This was also the motivation for the change of variables $r^2 = \rho^2 + 4m^2$ proposed by Chruściel [1] and already mentioned in footnote 1.

By associating these metric solutions under these two conditions, we would obtain a Wormhole and a White Fountain as a One-Way Membrane.

By associating these metric solutions under these two conditions, we would obtain a *Wormhole* and a *White Fountain* as a *One-Way Membrane*, connecting two semi-Riemannian spaces through a *"bridge"* that can be crossed only in one direction[4]. Let us assume further that the wormhole does not lead to

[4]The term *White Fountain* refers to a hypothetical construct where the bridge allows passage only in one direction, similar to the concept of a white hole in cosmology. This termi-

another universe as in Figure 1.a, or to a distant point in the same universe as in Figure 1.b; but that the two congruent sheets correspond to the same points in the physical universe through the transformation $u \to -u$ (or $\rho \to -\rho$), as suggested in [3] and in Section 3.3. We can then conclude that the two sheets are *PT-symmetric*[5].

In the literature, the inversion of the time coordinate has been analyzed in various ways. In particular:

(i) It was analyzed through the dynamic group theory of J-M Souriau ([17],[18]), and was shown to result in an inversion of energy. Consequently, time reversal transforms every motion of a particle of mass m into a motion of a particle of mass $-m$ ([18], page 191). On page 192 of the same book, the author offers an alternative analysis which avoids negative masses. Souriau points out that these alternatives should be judged according to their ability to explain experiments.

(ii) Feynman has offered an interpretation of antimatter as ordinary matter traveling backward in time.

(iii) It is known from theoretical analysis (the CPT theorem) and from experiments that elementary particles obey physical laws that are invariant under *CPT-symmetry*.

The *PT-symmetry* uncovered in Section 3 can be viewed as a *CPT-symmetry* followed by a *C-symmetry* (inversion of electric charge). This suggests that the *PT-symmetry* might lead to observable effects where particles and antiparticles are connected through the bridge. If the second sheet already contains ordinary matter, it could interact with the antimatter coming from the first sheet, constituting a potential source of energy. This provides a profound physical interpretation of the *PT-symmetry* in our geometric construction beyond the simple choice of coordinates.

5 The Bimetric Bridge

At the end of Section 3 we explained that according to [3], two points in the mathematical representation (6) with identical values of θ, ϕ but opposite values of u correspond to two points in physical space with the same value of r ($r = u^2 + m$). If we identify in (6) two points with opposite values of u, it seems that we are just left with a single sheet carrying the Schwarzschild solution in

nology is introduced to differentiate from traditional Einstein-Rosen bridges and to emphasize the unidirectional nature of the structure.

[5] This *PT-symmetry* is primarily a mathematical symmetry of the construction, representing how the two sheets mirror each other across the bridge. For physical spacetime, this symmetry indicates that the identified points on the two sheets correspond to the same physical location, but the interpretation requires careful consideration of the physical implications and potential observable effects.

the region $r > \alpha$. The "throat" $u = 0$ (or $r = \alpha$, in Schwarzschild coordinates) therefore appears as a limit of space rather than a gateway to a second sheet.

For the modified bridge studied in Section 3.1, the situation is more interesting because the two sheets carry different metrics (the ingoing and outgoing Eddington metrics). After identification of two points with opposite values of η in (13), we are again left with a single sheet but it is equipped with the two metrics:

$$\mathrm{d}s_+^2 = \left(1 - \frac{\alpha}{r}\right) c^2 \mathrm{d}t^2 - \left(1 + \frac{\alpha}{r}\right) \mathrm{d}r^2 - \frac{2\alpha c}{r} \mathrm{d}r\mathrm{d}t - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) \tag{21}$$

$$\mathrm{d}s_-^2 = \left(1 - \frac{\alpha}{r}\right) c^2 \mathrm{d}t^2 - \left(1 + \frac{\alpha}{r}\right) \mathrm{d}r^2 + \frac{2\alpha c}{r} \mathrm{d}r\mathrm{d}t - r^2 \left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2\right) \tag{22}$$

We therefore obtain a bimetric model. A particle that is infalling according to the first metric will reach the throat $r = \alpha$ for a finite value of t (say, $t = t_0$). For $t > t_0$ the second (outgoing) metric takes over. The particle will effectively seem to go back in time (it retraces its steps) since (22) is obtained from (21) by the transformation $t \mapsto -t$. This is consistent with the PT-symmetry uncovered in Section 3.1.

Comparison with Hossenfelder's bimetric theory. A "bimetric theory with exchange symmetry" was proposed by Hossenfelder in [8]. There are two types of matter in this theory, which can be viewed as matter of "positive mass" and "negative mass". Each type of matter follows the geodesics of its own metric. In Hossenfelder's theory, metric (21) describes the movement of positive masses in a field created by a point of positive mass since her theory reduces to General Relativity in this case. We therefore obtain the ordinary Schwarszchild metric (see equation (36) in [8]), or 21) in the ingoing Eddington coordinates. It is therefore natural to ask whether (22) might describe the movement of a particle of *negative* mass in the field created by the same positive positive mass as in (21). The answer to this question is negative because in Hossenfelder's theory, the corresponding metric is obtained from the ordinary Schwarszchild metric (8) by the transformation $\alpha \mapsto -\alpha$ (equation (37) in [8]). If we apply Eddington's change of variables to the resulting metric, we do not obtain anything like (22).[6]

6 Conclusion

This new geometric object behaves as a "One-Way Membrane", a union of a wormhole and a white fountain through a "bridge". The wormhole allows passage between two points, while the white fountain aspect ensures the unidirectional nature, preventing return through the bridge. We introduce a new geometric construction based on the static solution with spherical symmetry of

[6] As pointed out in Section V of [8], "changing to one of the more well-behaved systems with e.g. in-/outgoing Eddington-Finkelstein coordinates will be a nice transformation for the usual metric $\mathbf{g}$, but completely mess up the other metric $\underline{\mathbf{h}}$."

Einstein's equation in vacuum, by limiting ourselves to the only two hypotheses, inspired by physics: isotropy (invariance by $SO(3)$) and stationarity (invariance by time translation). This new geometric object behaves as a *"One-Way Membrane"*, a union of a wormhole and a white fountain through a *"bridge"*. The wormhole allows passage between two points, while the white fountain aspect ensures the unidirectional nature, preventing return through the *"bridge"*. With a Lorentzian metric at infinity, this structure connects two *PT-symmetric* enantiomorphic semi-Riemannian spaces. Therefore, this object corresponds to the two-sheets covering of a four-dimensional spacetime, presenting themselves as *PT-symmetric*, connected along a *"bridge"*. Taking our inspiration from Einstein and Rosen, we have suggested to represent a point in physical space by a pair of congruent points, one on each of the two sheets. We have shown that this identification of congruent points should lead to observable physical effects when an object crosses the space bridge between the two sheets. Finally, we have proposed a bimetric model to realize this identification, and we compared it to Hossenfelder's bimetric theory.

References

[1] Chruściel, P. T., *Lectures on Mathematical Relativity*, Lecture notes, revised and extended May 2008 following lectures on black holes in Marrakech, Beijing, July 2006, Mathematical Institute and Hertford College, Oxford, LMPT, Fédération Denis Poisson, Tours, accessed online at `https://homepage.univie.ac.at/piotr.chrusciel/papers/BeijingAll.pdf`, pages 77-78.

[2] Eddington A., *A comparison with Withehead's and Einstein's Formulae*, Nature, 1925.

[3] Einstein A., Rosen N., *The Particle Problem in the General Theory of Relativity*, Phys. Rev. 48, 73, July 1935.

[4] Flamm L., *Contributions to Einstein's theory of gravitation*, 1916.

[5] Fuller R.W., Wheeler J.A, *Causality and multiply connected space-time.* Physical Review, 1962.

[6] Guendelman, E., Kaganovich, A., Nissimov, E., Pacheva, S., *Spherically symmetric and rotating wormholes produced by lightlike branes*, International Journal of Modern Physics A 25 (07), 1405-1428, 2010.

[7] Guendelman, E., Nissimov, E., Pacheva, S., Stoilov, M., *Einstein-Rosen "bridge" revisited and lightlike thin-shell wormholes*, Bulgarian Journal of Physics 44:84–97, 2017.

[8] Hossenfelder, S., *Bimetric theory with exchange symmetry*, Physical Review D 78, 2008.

[9] Koiran P., *Infall Time in the Eddington-Finkelstein Metric, with Application do Einstein-Rosen Bridges*, Inter. Jr. of Mod. Phys. D, 14, August 2021.

[10] Kruskal, M. D., *"Maximal Extension of Schwarzschild Metric,"* Physical Review, Vol. 119, No. 5, 1960.

[11] Misner C. W., Thorne K.S., Wheeler J.A. *Gravitation*. Macmillan, 1973.

[12] Morris M., Thorne K.S., *"Wormholes in spacetime and their use for interstellar travel: A tool for teaching General Relativity"*, Am. J. Phys. 56, 395, 1988.

[13] Oppenheimer J.R., Snyder H., *On Continued Gravitational Contraction*, Phys. Rev. Vol. 56 pp 455-459, September 1939.

[14] Popławski, N., *Radial motion into an Einstein–Rosen bridge*, Physics Letters B 687:110–113, 2010.

[15] Schwarzschild K., *"Über das Gravitationsfeld Messenpunktes nach der Einsteinschen Theorie. Sit. Deut. Akad. Wiss.*, Januar 1916.

[16] Schwarzschild K., *"Über das Gravitionalsfeld einer Kugel Aus incompressibler Flüssigkeit nach der Einsteinschen Theorie. Sitzung der phys. Math. Klasse v.*, 23, Februar 1916.

[17] Souriau J. M., *Géométrie et relativité*, Hermann, (1964).

[18] Souriau J. M., *Structure of Dynamical Systems, a Symplectic View of Physics*, New York: Birkhäuser Verlag, (1997).

[19] Stoica O. C., *On singular semi-Riemannian manifolds*, International Journal of Geometric Methods in Modern Physics 11, 2014.

[20] Tolman R., Relativity, *Thermodynamics and Cosmology. The International Series of Monographs on Physics*, Oxford at the Clarendon Press. London, 1934.

[21] Wald R.M., *General Relativity*, University of Chicago Press, 1984.