Spin Particle Model between Two Half-Spaces of Minkowski Separated by an Interface

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Abstract

We develop the mathematical tools for a physical theory of relativity with positive and negative matter. We focus, like Jean Marie Souriau, on the action of the Poincaré group on the elements of the dual of the Lie algebra of the Poincaré group. We end the text with the study of two Minkowski half-spaces separated by an interface where an action of the Poincaré group occurs.

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Introduction

We will consider one type of objects in this text:

- **Spin particles**. The particles we will consider in this text are objects with essentially two characteristics: mass and spin. They are sensitive to the gravity of a body but are too small to produce spacetime deformation.

Thus, all the objects we consider are devoid of electric charge. Physical reality is certainly far from corresponding to this scheme, but we choose these limits.

In conventional physics, only objects with positive mass are considered. The idea of Jean-Pierre Petit (see [5], [6], [7], [8]) is to combine three existing models:

- the introduction of negative mass, as done by H. Bondi (see [1]) and W. Bonnor (see [2]);
- the introduction of a spacetime covering, as introduced by A. Sakharov (see [9], [10], [11]);
- the introduction of a general relativity model with two field equations, as done by S. Hossenfelder (see [3], [4]).

This Janus space is a two-sheeted spacetime covering, with positive masses living in one space and negative masses living in the other. Only gravitational interaction creates a link between these two spaces. This Janus universe can then be equipped with two field equations. More detailed information from Jean-Pierre Petit can be found in the book by Hicham Zejli (see [18]).

The purpose of this text is to define a "possible" mathematical framework for studying these ideas in affine case. We study the motion of a spin particle in a Minkowski space. The mass of a particle can be either a positive or negative real number. The guiding thread of this text is the book by Jean Marie Souriau [17].

This text is a compilation of notes I took while working on the Janus model. It is in no way intended to claim the work of Jean-Pierre Petit, but simply to assist those who would like to explore his research and offer a more mathematical perspective. This work is not original; the ideas and results are already present in the works of Jean-Marie Souriau and Jean-Pierre Petit.

1 Action of the Poincaré group

In this section, we study the action of the Poincaré group on the elements of the dual of its Lie algebra.

1.1 Lorentz Group

We begin by recalling the usual properties of the Lorentz group (see subsection B.1 for the definition of the τ maps).

Definition 1.1

The **Minkowski space** $\mathbb{R}^{1,3}$ is the space $\mathbb{R}^4$ equipped with the scalar product:

$$\eta := \mathrm{d}x^0 \otimes \mathrm{d}x^0 - \mathrm{d}x^1 \otimes \mathrm{d}x^1 - \mathrm{d}x^2 \otimes \mathrm{d}x^2 - \mathrm{d}x^3 \otimes \mathrm{d}x^3$$

i.e. we have :

$$\forall X, Y \in \mathbb{R}^4,\ \eta(X, Y) := \tau(X) Y.$$

Let $X, Y \in \mathbb{R}^{1,3}$. We say that X and Y are **orthogonal** if $\tau(X) Y = 0$. We then denote $X \perp_4 Y$.

By setting $(ct, x, y, z) := (x^0, x^1, x^2, x^3)$ (with c the speed of light), we have the usual form:

$$\eta := c^2 \mathrm{d}t \otimes \mathrm{d}t - \mathrm{d}x \otimes \mathrm{d}x - \mathrm{d}y \otimes \mathrm{d}y - \mathrm{d}z \otimes \mathrm{d}z.$$

We renormalize by assuming $c := 1$, we then obtain that:

$$\eta = \mathrm{d}t \otimes \mathrm{d}t - \mathrm{d}x \otimes \mathrm{d}x - \mathrm{d}y \otimes \mathrm{d}y - \mathrm{d}z \otimes \mathrm{d}z; \tag{1}$$

Let us set (see also subsection B.1):

$$I_{1,3} := (\eta_{ij})_{ij} := \begin{pmatrix} 1 & 0 \\ 0 & -I_3 \end{pmatrix}. \tag{2}$$

We thus have:

$$\eta = \sum_{i=1}^{4} \sum_{j=1}^{4} \eta_{ij} \mathrm{d}x^i \otimes \mathrm{d}x^i. \tag{3}$$

We then deduce the definition of the Lorentz group.

Definition 1.2

The **Lorentz group** is defined by:

$$\mathscr{L}or := \mathrm{O}(1,3,\mathbb{R}) := \left\{ \mathbf{L} \in \mathrm{GL}(4,\mathbb{R}),\ \forall X, Y \in \mathbb{R}^4,\ \eta(\mathbf{L}X, \mathbf{L}Y) = \eta(X, Y) \right\}.$$

The Lorentz group is the subgroup of $\mathrm{GL}(4,\mathbb{R})$ of automorphisms preserving the scalar product of $\mathbb{R}^{1,3}$. Let us use the applications τ defined in subsection B.1 of the appendix. We have for all $\mathbf{L} \in \mathrm{GL}(4,\mathbb{R})$:

$$\begin{aligned} \mathbf{L} \in \mathscr{L}or &\iff \forall X, Y \in \mathbb{R}^4,\ \eta(\mathbf{L}X, \mathbf{L}Y) = \eta(X, Y) \\ &\iff \forall X, Y \in \mathbb{R}^4,\ (\mathbf{L}X)^T I_{1,3} (\mathbf{L}Y) = X^T I_{1,3} Y \\ &\iff \mathbf{L}^T I_{1,3} \mathbf{L} = I_{1,3} \\ &\iff \tau(\mathbf{L})\mathbf{L} = I_4 \end{aligned}$$

Thus:

$$\mathscr{L}or = \left\{ \mathbf{L} \in \mathrm{GL}(4,\mathbb{R}),\ \mathbf{L}^T I_{1,3} \mathbf{L} = I_{1,3} \right\} = \left\{ \mathbf{L} \in \mathrm{GL}(4,\mathbb{R}),\ \tau(\mathbf{L})\mathbf{L} = I_4 \right\}. \tag{4}$$

We have the usual lemma.

Lemma 1.3

Let $\mathbf{L} \in \mathscr{L}or$. We have:

$$\det(\mathbf{L}) = \pm 1 \quad , \quad [\mathbf{L}]_{00}^2 \geq 1.$$

Proof. Since $\mathbf{L}^T I_{1,3} \mathbf{L} = I_{1,3}$, by taking the determinant we have $\det(\mathbf{L})^2 = 1$, hence $\det(\mathbf{L}) = \pm 1$. And since:

$$1 = [I_{1,3}]_{00} = [\mathbf{L}^T I_{1,3} \mathbf{L}]_{00} = \sum_{i=1}^{4} \sum_{j=1}^{4} [\mathbf{L}^T]_{0i} \eta_{ij} [\mathbf{L}]_{j0} = [\mathbf{L}]_{00}^2 - [\mathbf{L}]_{10}^2 - [\mathbf{L}]_{20}^2 - [\mathbf{L}]_{30}^2,$$

we necessarily have $[\mathbf{L}]_{00}^2 \geq 1$. □

The Lorentz group thus has four connected components:

$$\mathscr{L}or = \mathscr{L}or_n \sqcup \mathscr{L}or_s \sqcup \mathscr{L}or_t \sqcup \mathscr{L}or_{st}. \tag{5}$$

with:

- $\mathscr{L}or_n$ is the neutral component (its **restricted subgroup**), does not invert either space or time *i.e.* defined by:
$$\mathscr{L}or_n := \mathrm{SO}_o(1,3,\mathbb{R}) := \{\mathbf{L} \in \mathscr{L}or, \ \det(\mathbf{L}) = 1 \ \wedge \ [\mathbf{L}]_{00} \geq 1\}; \tag{6}$$
- $\mathscr{L}or_s$ inverts space *i.e.* defined by:
$$\mathscr{L}or_s := \{\mathbf{L} \in \mathscr{L}or, \ \det(\mathbf{L}) = -1 \ \wedge \ [\mathbf{L}]_{00} \geq 1\}; \tag{7}$$
- $\mathscr{L}or_t$ inverts time but not space *i.e.* defined by:
$$\mathscr{L}or_t := \{\mathbf{L} \in \mathscr{L}or, \ \det(\mathbf{L}) = 1 \ \wedge \ [\mathbf{L}]_{00} \leq -1\}; \tag{8}$$
- $\mathscr{L}or_{st}$ inverts both space and time *i.e.* defined by:
$$\mathscr{L}or_{st} := \{\mathbf{L} \in \mathscr{L}or, \ \det(\mathbf{L}) = -1 \ \wedge \ [\mathbf{L}]_{00} \leq -1\}. \tag{9}$$

The first two components are grouped together to form the subgroup called **orthochronous**:

$$\mathscr{L}or_o := \mathrm{SO}(1,3,\mathbb{R}) := \mathscr{L}or_n \sqcup \mathscr{L}or_s \tag{10}$$

The last two components form the subset antichronous, whose components invert time:

$$\mathscr{L}or_a := \mathscr{L}or_t \sqcup \mathscr{L}or_{st} \tag{11}$$

Thus, we have:

$$\mathscr{L}or = \mathscr{L}or_o \sqcup \mathscr{L}or_a \tag{12}$$

We will give another form to these connected components. Let:

$$\widetilde{\mathbf{T}} := -I_{1,3} \quad , \quad \widetilde{\mathbf{P}} := \widetilde{\mathbf{T}}. \tag{13}$$

Definition 1.4

(i) The **$\widetilde{\mathbf{P}}\widetilde{\mathbf{T}}$-group** is the subgroup $\widetilde{\mathscr{K}}$ of $\mathscr{L}or$ of order 4 generated by $\widetilde{\mathbf{P}}$ and $\widetilde{\mathbf{T}}$ *ie*:

$$\widetilde{\mathscr{K}} := \left\{\widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{P}}^{\lambda}, \ \nu, \lambda \in \{0,1\}\right\} = \left\{I_4, \widetilde{\mathbf{P}}, \widetilde{\mathbf{T}}, \widetilde{\mathbf{P}}\widetilde{\mathbf{T}}\right\}.$$

(ii) For all $\mathbf{B} \in \widetilde{\mathscr{K}}$, the **B-component of** $\mathscr{L}or$ is:

$$\mathscr{L}or\left(\mathbf{B}\right) := \left\{\mathbf{LB},\ \mathbf{L} \in \mathscr{L}or_n\right\}.$$

We have for all $\nu, \lambda \in \{0,1\}$:

$$\mathscr{L}or\left(\widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{T}}^{\lambda}\right) = \left\{\mathbf{L}_n\widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{T}}^{\lambda},\ \mathbf{L}_n \in \mathscr{L}or_n\right\}. \tag{14}$$

We then have a simple representation of the four components of $\mathscr{L}or$.

Lemma 1.5

The four connected components are:

$$\mathscr{L}or_n = \mathscr{L}or\left(\widetilde{\mathbf{P}}^{0}\widetilde{\mathbf{T}}^{0}\right) = \mathscr{L}or\left(I_4\right) \qquad \mathscr{L}or_s = \mathscr{L}or\left(\widetilde{\mathbf{P}}^{1}\widetilde{\mathbf{T}}^{0}\right) = \mathscr{L}or\left(\widetilde{\mathbf{P}}\right)$$

$$\mathscr{L}or_t = \mathscr{L}or\left(\widetilde{\mathbf{P}}^{0}\widetilde{\mathbf{T}}^{1}\right) = \mathscr{L}or\left(\widetilde{\mathbf{T}}\right) \qquad \mathscr{L}or_{st} = \mathscr{L}or\left(\widetilde{\mathbf{P}}^{1}\widetilde{\mathbf{T}}^{1}\right) = \mathscr{L}or\left(\widetilde{\mathbf{P}}\widetilde{\mathbf{T}}\right)$$

Proof. Equalities are shown by double inclusion. For example, let's demonstrate that $\mathscr{L}or_s = \mathscr{L}or\left(\widetilde{\mathbf{P}}\right)$. Take $\mathbf{L} \in \mathscr{L}or_s$ ($\det(\mathbf{L}) = 1$ and $[\mathbf{L}]_{00} \geq 1$). Then we have $\det(\mathbf{L}\widetilde{\mathbf{P}}) = -1$ and $[\mathbf{L}\widetilde{\mathbf{P}}]_{00} \geq 1$, hence we have $\mathbf{L}_n := \mathbf{L}\widetilde{\mathbf{P}} \in \mathscr{L}or_n$. Since $\widetilde{\mathbf{P}}^{-1} = \widetilde{\mathbf{P}}$, we can conclude that :

$$\mathbf{L} = \mathbf{L}_n\widetilde{\mathbf{P}} \in \mathscr{L}or\left(\widetilde{\mathbf{P}}\right).$$

The inclusion in the other direction is trivial. □

Thus, these 4 components are the 4 connected components of $\mathscr{L}or$, we have the decomposition:

$$\mathscr{L}or = \bigsqcup_{\mathbf{B} \in \widetilde{\mathscr{K}}} \mathscr{L}or\left(\mathbf{B}\right) = \bigsqcup_{\nu,\lambda \in \{0,1\}} \mathscr{L}or\left(\widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{T}}^{\lambda}\right). \tag{15}$$

1.2 Linear Torsors

Recall that there is a simple way to calculate the Lie algebra $\mathfrak{g}$ of a matrix Lie group G (i.e., a closed subgroup of $\mathrm{GL}(n,\mathbb{R})$ with $n \geq 1$) using the following equality:

$$\mathfrak{g} := \mathrm{T}_{I_n}G = \left\{M \in \mathscr{M}(n,\mathbb{R}),\ \forall t \in \mathbb{R},\ e^{tM} \in G\right\}. \tag{16}$$

From this, we can deduce the following result (see section B of the appendix for important results on the Lie algebra $(\mathscr{A}(1,3,\mathbb{R}),[])$).

Lemma 1.6

The group $\mathscr{L}or$ is a Lie group of dimension 6, and its Lie algebra is given by:

$$(\mathfrak{lor},[]) = (\mathscr{A}(1,3,\mathbb{R}),[]).$$

Proof. The group $\mathscr{L}or$ is closed in $\mathrm{GL}(4,\mathbb{R})$ because it is the preimage of the singleton $\{I_4\}$ under the continuous map $f : M \in \mathscr{M}(4,\mathbb{R}) \longrightarrow \tau(M)M$. By equation (16), the Lie algebra $\mathfrak{lor}$ of $\mathscr{L}or$ is given by:

$$\begin{aligned}\mathfrak{lor} :=& \mathrm{T}_{I_4}\mathrm{O}(1,3,\mathbb{R}) = \left\{M \in \mathscr{M}(4,\mathbb{R}),\ \forall t \in \mathbb{R},\ e^{tM} \in \mathrm{O}(1,3,\mathbb{R})\right\}\\ &= \left\{M \in \mathscr{M}(4,\mathbb{R}),\ \forall t \in \mathbb{R},\ \tau\left(e^{tM}\right)e^{tM} = I_4\right\}\end{aligned}$$

Let us then define for any $M \in \mathscr{M}(4,\mathbb{R})$ the smooth map from $\mathbb{R}$ to $\mathrm{GL}(4,\mathbb{R})$:

$$g_M : t \in \mathbb{R} \longmapsto \tau\left(e^{tM}\right) e^{tM}.$$

Let $M \in \mathscr{M}(4,\mathbb{R})$. For all $t \in \mathbb{R}$, we have:

$$g_M(t) = I_{1,3}\left(e^{tM}\right)^T I_{1,3} e^{tM} = e^{t(I_{1,3}M^T I_{1,3}+M)} = e^{t(\tau(M)+M)}$$

Since $g'_M : t \in \mathbb{R} \longmapsto (\tau(M)+M)e^{t(\tau(M)+M)}$, we then have the equivalences:

$$\begin{aligned} M \in \mathfrak{lor} &\iff \forall t \in \mathbb{R},\ g_M(t) = I_4 \\ &\iff \forall t \in \mathbb{R},\ g'_M(t) = 0 \ \wedge\ g_M(0) = I_4 \\ &\iff \tau(M) + M = 0 \\ &\iff M \in \mathscr{A}(1,3,\mathbb{R}) \end{aligned}$$

□

Thus, $\mathfrak{lor}$ has the basis (see equation (97)) :

$$\mathscr{B}ase\,(\mathfrak{lor}) = \left\{ e_i \ominus 0 := \begin{pmatrix} 0 & e_i^T \\ e_i & 0 \end{pmatrix},\ i \in \{1,2,3\} \right\} \bigsqcup \left\{ 0 \ominus e_i := \begin{pmatrix} 0 & 0 \\ 0 & j(e_i) \end{pmatrix},\ i \in \{1,2,3\} \right\}. \tag{17}$$

Similarly, $\mathscr{L}or_o := \mathrm{SO}(1,3,\mathbb{R})$ is an open subset of $\mathrm{O}(1,3,\mathbb{R})$ because it is the preimage of the open set $\{1\}$ under the continuous map $\det : \mathrm{O}(1,3,\mathbb{R}) \longrightarrow \{\pm 1\}$. Thus, the Lie algebra $\mathfrak{lor}_o$ of $\mathscr{L}or_o$ equals $\mathfrak{lor}$.

We have the characterization of the dual of $\mathfrak{lor}$ (see lemma B.6):

$$\mathfrak{lor}^* = \mathscr{A}(1,3,\mathbb{R})^* = \left\{ \{\ M\ \} : \Lambda \longmapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda),\ M \in \mathscr{A}(1,3,\mathbb{R}) \right\} \tag{18}$$

Definition 1.7

(i) The elements of $\mathfrak{lor}^*$ are called **linear torsors**.

(ii) Let $\widetilde{\mu} := \{\ M\ \} \in \mathfrak{lor}^*$. The matrix $M(\widetilde{\mu}) := M \in \mathscr{A}(1,3,\mathbb{R})$ is called the **moment matrix associated with** $\widetilde{\mu}$.

Let Ad^* denote the coadjoint representation on $\mathfrak{lor}^*$:

$$\begin{array}{rccc} \mathrm{Ad}^* : & \mathfrak{lor} & \longrightarrow & \mathrm{Aut}(\mathfrak{lor}^*) \\ & \mathbf{L} & \longmapsto & \mathrm{Ad}^*_{\mathbf{L}} : \widetilde{\mu} \longmapsto \left(\Lambda \longmapsto \widetilde{\mu}\left(\mathbf{L}^{-1}\Lambda\mathbf{L}\right)\right) \end{array} \tag{19}$$

Definition 1.8

The **action of the group** $\mathscr{L}or$ **on** $\mathfrak{lor}^*$ is defined by the coadjoint representation, that is, for any $\mathbf{L} \in \mathscr{L}or$ and any $\widetilde{\mu} \in \mathfrak{lor}^*$, we denote this action by:

$$\mathbf{L} \bullet \widetilde{\mu} := \mathrm{Ad}^*_{\mathbf{L}}(\widetilde{\mu}).$$

We have a simple description of this action on the torsors.

Proposition 1.9

Let $\mathbf{L} \in \mathscr{L}or$ and $\left\{ \ M \ \right\} \in \mathfrak{lor}^*$. We have:

$$\mathbf{L} \bullet \left\{ \ M \ \right\} = \left\{ \ \mathbf{L} M \tau(\mathbf{L}) \ \right\}.$$

Proof. We have:

$$\left(\mathbf{L} \bullet \left\{ \ M \ \right\}\right) \Lambda = \left\{ \ M \ \right\} (\tau(\mathbf{L}) \Lambda \mathbf{L}) = -\frac{1}{2} \mathrm{Tr}\left(M \tau(\mathbf{L}) \Lambda \mathbf{L}\right) = -\frac{1}{2} \mathrm{Tr}\left(\mathbf{L} M \tau(\mathbf{L}) \Lambda\right) = \left\{ \ \mathbf{L} M \tau(\mathbf{L}) \ \right\} \Lambda$$

and we have $\mathbf{L} M \tau(\mathbf{L}) \in \mathscr{A}(1,3,\mathbb{R})$ because:

$$\tau(\mathbf{L} M \tau(\mathbf{L})) = \mathbf{L} \tau(M) \tau(\mathbf{L}) = -\mathbf{L} M \tau(\mathbf{L}).$$

□

From proposition 1.9, we deduce the following corollary.

Corollary 1.10

Let $\mathbf{L} := \mathbf{L}_n \widetilde{\mathbf{P}}^\nu \widetilde{\mathbf{T}}^\lambda \in \mathscr{L}or$ and $\widetilde{\mu} \in \mathfrak{lor}^*$. We have:

$$M(\mathbf{L} \bullet \widetilde{\mu}) = \mathbf{L} M(\widetilde{\mu}) \tau(\mathbf{L})$$
$$M\left(\left(\mathbf{L}_n \widetilde{\mathbf{P}}^\nu \widetilde{\mathbf{T}}^\lambda\right) \bullet \widetilde{\mu}\right) = \mathbf{L}_n \widetilde{\mathbf{P}}^\nu \widetilde{\mathbf{T}}^\lambda M \widetilde{\mathbf{T}}^\lambda \widetilde{\mathbf{P}}^\nu \tau(\mathbf{L}_n)$$

To describe the Lie algebra of $\mathscr{L}or$, we can also use the isomorphism of Lie algebras (see subsection A.4 of the appendix for more details):

$$j : (\mathbb{R}^3, \wedge) \longmapsto (\mathscr{A}(3,\mathbb{R}), [\ ,\ ])$$

Thus, we have:

$$\mathfrak{lor} = \mathscr{A}(1,3,\mathbb{R}) = \left\{ \begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix}, \ \beta, w \in \mathbb{R}^3 \right\}. \tag{20}$$

Therefore, for all $\left\{ \ M \ \right\} \in \mathfrak{lor}^*$ and any $\Lambda \in \mathfrak{lor}$, there are unique $\ell, g, \beta, w \in \mathbb{R}^3$ such that:

$$\left\{ \ M \ \right\} \Lambda = \left\{ \ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \ \right\} \begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix} = -\frac{1}{2} \mathrm{Tr}\left(\begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix} \right) = \ell^T w - g^T \beta \tag{21}$$

We denote this last equality as:

$$\left\{ \ \ell \mid g \ \right\} \begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix}. \tag{22}$$

The dual $\mathfrak{lor}^*$ has the following description:

$$\mathfrak{lor}^* = \left\{ \left\{ \ \ell \mid g \ \right\} : \begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix} \longmapsto \ell^T w - g^T \beta, \ \ell, g \in \mathbb{R}^3 \right\}. \tag{23}$$

We have the following definitions.

Definition 1.11

Let

$$\widetilde{\mu} := \left\{ \ M \ \right\} := \left\{ \ \ell \mid g \ \right\} \in \mathfrak{poin}^*$$

with the relation:

$$M = \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix}.$$

(i) The vector $\ell := \ell(\widetilde{\mu}) \in \mathbb{R}^3$ is called the **angular momentum of M associated with $\widetilde{\mu}$.**

(ii) The vector $g := g(\widetilde{\mu}) \in \mathbb{R}^3$ is the **relativistic barycenter of M associated with $\widetilde{\mu}$.**

Proposition 1.12

Let $\{\ \ell \mid g\ \} \in \mathfrak{lor}^*$ and $\lambda, \nu \in \{0,1\}$. We have:

$$(\widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{T}}^{\lambda}) \bullet \{\ \ell \mid g\ \} = \{\ l \mid (-1)^{\lambda+\nu} g\ \}$$

Proof. Let us set $\widetilde{\mu} := \{\ M\ \} := \{\ \ell \mid g\ \} \in \mathfrak{lor}^*$. By corollary 1.10, we have :

$$M((\widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{T}}^{\lambda}) \bullet \widetilde{\mu}) = (\widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{T}}^{\lambda}) M(\widetilde{\mu}) \tau(\widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{T}}^{\lambda}) = (\widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{T}}^{\lambda}) M(\widetilde{\mu}) \widetilde{\mathbf{T}}^{\lambda}\widetilde{\mathbf{P}}^{\nu}$$

Thus we have:

$$\begin{aligned}
(\widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{T}}^{\lambda}) \bullet \{\ \ell \mid g\ \} &= (\widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{T}}^{\lambda}) \bullet \left\{ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \right\} \\
&= \left\{ \widetilde{\mathbf{P}}^{\nu}\widetilde{\mathbf{T}}^{\lambda} \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \widetilde{\mathbf{T}}^{\lambda}\widetilde{\mathbf{P}}^{\nu} \right\} \\
&= \left\{ \begin{pmatrix} 0 & (-1)^{\lambda+\nu} g^T \\ (-1)^{\lambda+\nu} g & j(\ell) \end{pmatrix} \right\} \\
&= \{\ l \mid (-1)^{\lambda+\nu} g\ \}
\end{aligned}$$

□

We deduce a simple expression of the action of the **PT**-group $\mathscr{K}$ on the torsors of $\mathfrak{lor}^*$. For any $\{\ \ell \mid g\ \} \in \mathfrak{lor}^*$, we have:

$$\widetilde{\mathbf{P}} \bullet \{\ \ell \mid g\ \} = \{\ \ell \mid -g\ \} \tag{24}$$
$$\widetilde{\mathbf{T}} \bullet \{\ \ell \mid g\ \} = \{\ \ell \mid -g\ \} \tag{25}$$

1.3 Poincaré Group

We define the Poincaré group as the group of isometries of Minkowski space. Mathematically, this means that it is the affine group $\mathrm{Aff}(\mathscr{L}or)$ associated with the Lorentz group $\mathscr{L}or$.

Definition 1.13: Associated Affine Group

The **Poincaré group** is defined by:

$$\mathscr{P}oin := \mathrm{Aff}(\mathscr{L}or) := \mathscr{L}or \ltimes \mathbb{R}^{1,3}.$$

For all $(\mathbf{L}, C), (\mathbf{L}', C')$, the composition law on $\mathscr{P}oin$ is defined by:

$$(\mathbf{L}, C) \cdot (\mathbf{L}', C') := (\mathbf{L}\mathbf{L}', C + \mathbf{L}C'). \tag{26}$$

The action of an element $\mathbf{A} := (\mathbf{L}, C) \in \mathscr{P}oin$ on an element $X \in \mathbb{R}^{1,3}$ is given by:

$$\mathbf{A} \bullet X = \mathbf{L}X + C. \tag{27}$$

We will revisit this action later with the matrix representation of the elements of $\mathscr{P}oin$.

1.4 Lie Algebra

Let $\mathfrak{poin}$ be the Lie algebra of the Poincaré group. The Lie algebra $\mathfrak{poin}$ is the vector space product $\mathscr{A}(1,3,\mathbb{R}) \times \mathbb{R}^4$ equipped with the Lie bracket[1]:

$$[(\Lambda,\Gamma),(\Lambda',\Gamma')] := ([\Lambda,\Lambda'],\Lambda\Gamma' - \Lambda'\Gamma). \tag{28}$$

It has dimension $10 = 6 + 4$ over $\mathbb{R}$, and a basis is given by:

$$\mathscr{B}ase\,(\mathfrak{poin}) = \{(e_i \ominus 0, 0),\ i \in \{1,2,3\}\} \bigsqcup \{(0 \ominus e_i, 0),\ i \in \{1,2,3\}\} \bigsqcup \left\{(0, e_i^{(4)}),\ i \in \{1,2,3,4\}\right\} \tag{29}$$

with:

$$\mathscr{B}ase\left(\mathbb{R}^4\right) := \left(e_1^{(4)} := \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, e_2^{(4)} := \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, e_3^{(4)} := \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, e_4^{(4)} := \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right). \tag{30}$$

1.5 Matrix Representation

We will use a matrix representation of the group $\mathscr{P}oin$ and its Lie algebra through the following embedding.

Lemma 1.14

Let the application be:

$$\begin{array}{rccc} \Psi : & (\mathscr{P}oin, \cdot) & \longrightarrow & (\mathrm{GL}(5,\mathbb{R}), \times) \\ & (\mathbf{L}, C) & \longmapsto & \begin{pmatrix}\mathbf{L} & C\\ 0 & 1\end{pmatrix} \end{array}$$

Then Ψ is an injective group morphism.

Proof. The application is clearly injective, and for all $(U,D),(U',D') \in \mathscr{P}oin$, we have:

$$\Psi(\mathbf{L},C)\Psi(\mathbf{L}',C') = \begin{pmatrix}\mathbf{L} & C\\ 0 & 1\end{pmatrix}\begin{pmatrix}\mathbf{L}' & C'\\ 0 & 1\end{pmatrix} = \begin{pmatrix}\mathbf{L}\mathbf{L}' & C + \mathbf{L}C'\\ 0 & 1\end{pmatrix} = \Psi(\mathbf{L}\mathbf{L}', C + \mathbf{L}C') = \Psi((\mathbf{L},C)\cdot(\mathbf{L}',C'))$$

Thus, the result follows. □

Thus, we can identify $(\mathscr{P}oin, \cdot)$ as a subgroup of $(\mathrm{GL}(5,\mathbb{R}), \times)$. This is what we will do from now on, i.e.:

$$\mathscr{P}oin := \left\{ \begin{pmatrix}\mathbf{L} & C\\ 0 & 1\end{pmatrix},\ \mathbf{L} \in \mathscr{L}or\ \wedge\ C \in \mathbb{R}^{1,3} \right\}. \tag{31}$$

$\mathscr{P}oin$ is therefore a Lie subgroup of the group $\mathrm{GL}(5,\mathbb{R})$. For example, we have:

$$\forall \mathbf{A} := \begin{pmatrix}\mathbf{L} & C\\ 0 & 1\end{pmatrix} \in \mathscr{P}oin,\ \mathbf{A}^{-1} = \begin{pmatrix}\tau(\mathbf{L}) & -\tau(\mathbf{L})C\\ 0 & 1\end{pmatrix}. \tag{32}$$

The action of an element $\mathbf{A} := \begin{pmatrix}\mathbf{L} & C\\ 0 & 1\end{pmatrix} \in \mathscr{P}oin$ on an element $X \in \mathbb{R}^{1,3}$ is given, as in equation (27), by:

$$\mathbf{A} \bullet X := \begin{pmatrix}\mathbf{L} & C\\ 0 & 1\end{pmatrix}\begin{pmatrix}X\\ 1\end{pmatrix} = \mathbf{L}X + C. \tag{33}$$

By equation (16) and since the exponential of any matrix is invertible, the Lie algebra $\mathfrak{gl}(5,\mathbb{R})$ of $\mathrm{GL}(5,\mathbb{R})$ is simply given by:

$$\mathfrak{gl}(5,\mathbb{R}) := \mathrm{T}_{I_4}\mathrm{GL}(5,\mathbb{R}) = \left\{ M \in \mathscr{M}(4,\mathbb{R}),\ \forall t \in \mathbb{R},\ e^{tM} \in \mathrm{GL}(5,\mathbb{R}) \right\} = \mathscr{M}(5,\mathbb{R}). \tag{34}$$

Thus, from the natural embedding Ψ, we deduce the following embedding.

[1] Do not confuse $\mathfrak{poin}$ with the product Lie algebra $\mathfrak{lor} \times \mathbb{R}^4$ equipped with the trivial Lie bracket $[(\Lambda,\Gamma),(\Lambda',\Gamma')] := ([\Lambda,\Lambda'], 0)$.

Lemma 1.15

We have a natural embedding of Lie algebras:

$$\begin{array}{rccc} \psi : & (\mathfrak{poin}, []) & \longrightarrow & (\mathfrak{gl}(5,\mathbb{R}), []) \\ & (\Lambda, \Gamma) & \longmapsto & \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \end{array}$$

Proof. For all $(\Lambda, \Gamma), (\Lambda', \Gamma') \in \mathfrak{poin}$, we have:

$$\begin{aligned} \big[(\Lambda,\Gamma),(\Lambda',\Gamma')\big] &= \left[\psi^{-1}\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix}, \psi^{-1}\begin{pmatrix} \Lambda' & \Gamma' \\ 0 & 0 \end{pmatrix}\right] \\ &= \psi^{-1}\left(\left[\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \Lambda' & \Gamma' \\ 0 & 0 \end{pmatrix}\right]\right) \\ &= \psi^{-1}\left(\left[\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} \Lambda' & \Gamma' \\ 0 & 0 \end{pmatrix}\right]\right) \\ &= \psi^{-1}\left(\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Lambda' & \Gamma' \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \Lambda' & \Gamma' \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix}\right) \\ &= \psi^{-1}\begin{pmatrix} \Lambda\Lambda' - \Lambda'\Lambda & \Lambda\Gamma' - \Lambda'\Gamma \\ 0 & 0 \end{pmatrix} \\ &= (\Lambda\Lambda' - \Lambda'\Lambda, \Lambda\Gamma' - \Lambda'\Gamma) \end{aligned}$$

□

We then identify the Lie algebra $\mathfrak{poin}$ with its image $\psi(\mathfrak{poin})$ via ψ, and the bracket on $\psi(\mathfrak{poin})$ is derived from the usual Lie bracket on $\mathfrak{gl}(5,\mathbb{R})$. We thus have an isomorphism of Lie algebras, also denoted ψ:

$$\begin{array}{rccc} \psi : & (\mathfrak{poin}, []) & \longrightarrow & (\psi(\mathfrak{poin}), []) \\ & (\Lambda, \Gamma) & \longmapsto & \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \end{array}$$

Thus, we can identify $(\mathfrak{poin}, [])$ as the Lie subalgebra of $(\mathfrak{gl}(5,\mathbb{R}), [])$:

$$\mathfrak{poin} = \left\{\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix},\ \Lambda \in \mathfrak{lor}\ \wedge\ \Gamma \in \mathbb{R}^{1,3}\right\}. \tag{35}$$

with a basis (see also (29)):

$$\mathscr{B}ase\,(\mathfrak{poin}) = \left\{\begin{pmatrix} e_i \ominus 0 & 0 \\ 0 & 0 \end{pmatrix},\ i \in \{1,2,3\}\right\} \bigsqcup \left\{\begin{pmatrix} 0 \ominus e_i & 0 \\ 0 & 0 \end{pmatrix},\ i \in \{1,2,3\}\right\} \bigsqcup \left\{\begin{pmatrix} 0 & e_i^{(4)} \\ 0 & 0 \end{pmatrix},\ i \in \{1,2,3,4\}\right\}. \tag{36}$$

We now define the connected components similar to those of the Lorentz group. The **restricted Poincaré group** is the subgroup of $\mathscr{P}oin$ given by:

$$\mathscr{P}oin_n := \left\{\begin{pmatrix} \mathbf{L}_n & C \\ 0 & 1 \end{pmatrix},\ \mathbf{L}_n \in \mathscr{L}or_n\ \wedge\ C \in \mathbb{R}^{1,3}\right\}. \tag{37}$$

Let us set:

$$\mathbf{P} := \begin{pmatrix} \widetilde{\mathbf{P}} & 0 \\ 0 & 1 \end{pmatrix} \quad , \quad \mathbf{T} := \begin{pmatrix} \widetilde{\mathbf{T}} & 0 \\ 0 & 1 \end{pmatrix}. \tag{38}$$

We have:

$$\forall \begin{pmatrix} \mathbf{L}_n & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin_n,\ \forall \lambda, \nu \in \{0,1\},\ \begin{pmatrix} \mathbf{L}_n & C \\ 0 & 1 \end{pmatrix} \mathbf{P}^\nu \mathbf{T}^\lambda = \begin{pmatrix} \mathbf{L}_n \widetilde{\mathbf{P}}^\nu \widetilde{\mathbf{T}}^\lambda & C \\ 0 & 1 \end{pmatrix}$$

and therefore by equation (15):

$$\mathscr{P}oin = \left\{ \begin{pmatrix} \mathbf{L}_n \widetilde{\mathbf{P}}^{\nu} \widetilde{\mathbf{T}}^{\lambda} & C \\ 0 & 1 \end{pmatrix}, \ \lambda, \nu \in \{0,1\} \ \wedge \ \mathbf{L}_n \in \mathscr{L}or_n \ \wedge \ C \in \mathbb{R}^{1,3} \right\}. \tag{39}$$

Definition 1.16

(i) The **PT-group** is the subgroup $\mathscr{K}$ of $\mathscr{P}oin$ of order 4 generated by $\mathbf{P}$ and $\mathbf{T}$, i.e.:

$$\mathscr{K} := \left\{ \mathbf{P}^{\nu}\mathbf{T}^{\lambda}, \ \nu, \lambda \in \{0,1\} \right\} = \{I_5, \mathbf{P}, \mathbf{T}, \mathbf{PT}\}.$$

(ii) For all $\mathbf{B} \in \mathscr{K}$, the **B-component of** $\mathscr{P}oin$ is:

$$\mathscr{P}oin\,(\mathbf{B}) := \{\mathbf{AB}, \ \mathbf{A} \in \mathscr{P}oin_n\}.$$

Thus, for all $\nu, \lambda \in \{0,1\}$:

$$\mathscr{P}oin\left(\mathbf{P}^{\nu}\mathbf{T}^{\lambda}\right) = \left\{ \begin{pmatrix} \mathbf{L}_n \widetilde{\mathbf{P}}^{\nu} \widetilde{\mathbf{T}}^{\lambda} & C \\ 0 & 1 \end{pmatrix}, \ \mathbf{L}_n \in \mathscr{L}or_n \ \wedge \ C \in \mathbb{R}^{1,3} \right\}.$$

These 4 components are the 4 connected components of $\mathscr{P}oin$. We have the decomposition:

$$\mathscr{P}oin = \bigsqcup_{\mathbf{B} \in \mathscr{K}} \mathscr{P}oin\,(\mathbf{B}) = \bigsqcup_{\nu,\lambda \in \{0,1\}} \mathscr{P}oin\left(\mathbf{P}^{\nu}\mathbf{T}^{\lambda}\right). \tag{40}$$

As for the Lorentz group, we have the **orthochronous Poincaré subgroup** defined by:

$$\mathscr{P}oin_o := \mathscr{P}oin\,(I_5) \ \sqcup \ \mathscr{P}oin\,(\mathbf{P}) \tag{41}$$

and the subset of elements that reverse time, the **antichronous part** defined by:

$$\mathscr{P}oin_a := \mathscr{P}oin\,(\mathbf{T}) \ \sqcup \ \mathscr{P}oin\,(\mathbf{PT}) \tag{42}$$

Thus, we have:

$$\mathscr{P}oin = \mathscr{P}oin_o \ \sqcup \ \mathscr{P}oin_a \tag{43}$$

1.6 Action on Tensors

We have the following representation of the dual of the Lie algebra of the Poincaré group.

Corollary 1.17

We have:

$$\mathfrak{poin}^* = \left\{ \{\ M \mid P\ \} : \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \longmapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda) - \tau(P)\Gamma, \ M \in \mathscr{A}(1,3,\mathbb{R}) \ \wedge \ P \in \mathbb{R}^{1,3} \right\}.$$

Proof. This follows from the representation of $\mathfrak{lor}$ (see lemma B.6) and from the characterization:

$$\left(\mathbb{R}^{1,3}\right)^* = \left\{\Gamma \longmapsto -\tau(P)\Gamma, \ P \in \mathbb{R}^{1,3}\ \right\}.$$

□

Definition 1.18

(i) The elements of $\mathfrak{poin}^*$ are called **affine torsors**.

(ii) Let
$$\mu := \{ \ M \mid P \ \} \in \mathfrak{poin}^*.$$

(a) The matrix $M(\mu) := M \in \mathscr{A}(1,3,\mathbb{R})$ is called the **moment matrix associated with** μ.

(b) The vector $P(\mu) := P \in \mathbb{R}^{1,3}$ is called the **stress–energy vector associated with** μ.

Let us denote Ad^* the coadjoint representation on $\mathfrak{poin}^*$:

$$\begin{array}{rccc} \mathrm{Ad}^* : & \mathscr{P}oin & \longrightarrow & \mathrm{Aut}(\mathfrak{poin}^*) \\ & \mathbf{A} & \longmapsto & \mathrm{Ad}^*_{\mathbf{A}} : \mu \longmapsto \left(Z \longmapsto \mu\left(\mathbf{A}^{-1} Z \mathbf{A}\right)\right) \end{array} \tag{44}$$

Definition 1.19

The **action of the group** $\mathscr{P}oin$ **on** $\mathfrak{poin}^*$ is defined by the coadjoint representation, i.e., for any $\mathbf{A} \in \mathscr{P}oin$ and any $\mu \in \mathfrak{poin}^*$, we denote this action by:

$$\mathbf{A} \bullet \mu := \mathrm{Ad}^*_{\mathbf{A}}(\mu).$$

We then have a simple description of this action on tensors.

Proposition 1.20

Let:
$$\mathbf{A} := \begin{pmatrix} \mathbf{L} & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin \quad , \quad \{ \ M \mid P \ \} \in \mathfrak{poin}^*.$$

We have:

$$\mathbf{A} \bullet \{ \ M \mid P \ \} = \{ \ \mathbf{L} M \tau(\mathbf{L}) + C\tau(P)\tau(\mathbf{L}) - \mathbf{L}P\tau(C) \mid \mathbf{L}P \ \}.$$

Proof. We have:

$$\begin{aligned}
\left(\mathbf{A} \bullet \{ \ M \mid P \ \}\right)\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} &= \{ \ M \mid P \ \}\left(\mathbf{A}^{-1}\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix}\mathbf{A}\right) \\
&= \{ \ M \mid P \ \}\left(\begin{pmatrix} \tau(\mathbf{L}) & -\tau(\mathbf{L})C \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \mathbf{L} & C \\ 0 & 1 \end{pmatrix}\right) \\
&= \{ \ M \mid P \ \}\begin{pmatrix} \tau(\mathbf{L})\Lambda\mathbf{L} & \tau(\mathbf{L})(\Lambda C + \Gamma) \\ 0 & 0 \end{pmatrix} \\
&= -\frac{1}{2}\mathrm{Tr}\left(M\tau(\mathbf{L})\Lambda\mathbf{L}\right) - \tau(P)\tau(\mathbf{L})(\Lambda C + \Gamma) \\
&= -\frac{1}{2}\mathrm{Tr}\left[\left(\mathbf{L}M\tau(\mathbf{L}) + 2C\tau(P)\tau(\mathbf{L})\right)\Lambda\right] + \tau\left(\mathbf{L}P\right)\Gamma \\
&= -\frac{1}{2}\mathrm{Tr}\left[\left(\mathbf{L}M\tau(\mathbf{L}) + C\tau(P)\tau(\mathbf{L}) - \mathbf{L}P\tau(C)\right)\Lambda\right] + \tau\left(\mathbf{L}P\right)\Gamma \\
&= \{ \ \mathbf{L}M\tau(\mathbf{L}) + C\tau(P)\tau(\mathbf{L}) - \mathbf{L}P\tau(C) \mid \mathbf{L}P \ \}\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix}
\end{aligned}$$

and we indeed have $\mathbf{L}M\tau(\mathbf{L}) + C\tau(P)\tau(\mathbf{L}) - \mathbf{L}P\tau(C)\tau(\mathbf{L}) \in \mathscr{A}(1,3,\mathbb{R})$ because:

$$\tau(\mathbf{L}M\tau(\mathbf{L}) + C\tau(P)\tau(\mathbf{L}) - \mathbf{L}P\tau(C)\tau(\mathbf{L})) = \mathbf{L}\tau(M)\tau(\mathbf{L}) - \mathbf{L}P\tau(C) + C\tau(P)\tau(\mathbf{L})$$

$$= -\left(\mathbf{L}M\tau(\mathbf{L}) + C\tau(P)\tau(\mathbf{L}) - \mathbf{L}P\tau(C)\tau(\mathbf{L})\right).$$

□

We deduce from Proposition 1.20 the following corollary.

Corollary 1.21

For:

$$\mathbf{A} := \begin{pmatrix} \mathbf{L} & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin \quad , \quad \mu \in \mathfrak{poin}^*,$$

we have:

$$\begin{aligned} M(\mathbf{A} \bullet \mu) &= \mathbf{L}M(\mu)\tau(\mathbf{L}) + C\tau(P(\mu))\tau(\mathbf{L}) - \mathbf{L}P(\mu)\tau(C) \\ P(\mathbf{A} \bullet \mu) &= \mathbf{L}P(\mu) \end{aligned}$$

For:

$$\mathbf{A} := \begin{pmatrix} \mathbf{L}_n \widetilde{\mathbf{P}}^\nu \widetilde{\mathbf{T}}^\lambda & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin \quad , \quad \{\ M \mid P\ \} \in \mathfrak{poin}^*,$$

we have:

$$M(\mathbf{A} \bullet \mu) = \mathbf{L}_n \widetilde{\mathbf{P}}^\nu \widetilde{\mathbf{T}}^\lambda M \widetilde{\mathbf{T}}^\lambda \widetilde{\mathbf{P}}^\nu \tau(\mathbf{L}_n) + C\tau(P)\widetilde{\mathbf{T}}^\lambda \widetilde{\mathbf{P}}^\nu \tau(\mathbf{L}_n) - \mathbf{L}_n \widetilde{\mathbf{P}}^\nu \widetilde{\mathbf{T}}^\lambda P\tau(C) \tag{45}$$

$$P(\mathbf{A} \bullet \mu) = \mathbf{L}_n \widetilde{\mathbf{P}}^\nu \widetilde{\mathbf{T}}^\lambda P. \tag{46}$$

1.7 Second Representation of Tensors

As with the Lie algebra of the Lorentz group, to describe the Lie algebra of $\mathscr{P}oin$, we can also use the isomorphism of Lie algebras (see subsection A.4 of the appendix):

$$j : (\mathbb{R}^3, \wedge) \longleftrightarrow (\mathscr{A}(3,\mathbb{R}), [\ ,\ ])$$

Thus, we have:

$$\mathfrak{poin} = \left\{ \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix},\ \Lambda \in \mathscr{A}(1,3,\mathbb{R}) \ \wedge\ \Gamma \in \mathbb{R}^{1,3} \right\} = \left\{ \begin{pmatrix} 0 & \beta^T & \alpha \\ \beta & j(w) & \gamma \\ 0 & 0 & 0 \end{pmatrix},\ \beta, w, \gamma \in \mathbb{R}^3 \ \wedge\ \alpha \in \mathbb{R} \right\}. \tag{47}$$

Therefore, for all $\{\ M \mid P\ \} \in \mathfrak{poin}^*$ and for any $\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \in \mathfrak{lor}$, there are unique $\ell, g, p, \beta, w, \gamma \in \mathbb{R}^3$ and $E, \alpha \in \mathbb{R}$ such that:

$$\begin{aligned} \{\ M \mid P\ \} \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} &= \left\{ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \middle| \begin{pmatrix} E \\ p \end{pmatrix} \right\} \begin{pmatrix} 0 & \beta^T & \alpha \\ \beta & j(w) & \gamma \\ 0 & 0 & 0 \end{pmatrix} \\ &= -\frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \begin{pmatrix} 0 & \beta^T \\ \beta & j(w) \end{pmatrix} \right) - \begin{pmatrix} E & p^T \end{pmatrix} I_{1,3} \begin{pmatrix} \alpha \\ \gamma \end{pmatrix} \\ &= \ell^T w - g^T \beta + p^T \gamma - E\alpha \end{aligned}$$

We denote this last equality as:

$$\{\ \ell \mid g \mid p \mid E\ \} \begin{pmatrix} 0 & \beta^T & \alpha \\ \beta & j(w) & \gamma \\ 0 & 0 & 0 \end{pmatrix}. \tag{48}$$

The dual $\mathfrak{poin}^*$ has the following descriptions:

$$\mathfrak{poin}^* = \left\{ \{\ \ell \mid g \mid p \mid E\ \} : \begin{pmatrix} 0 & \beta^T & \alpha \\ \beta & j(w) & \gamma \\ 0 & 0 & 0 \end{pmatrix} \longmapsto \ell^T w - g^T \beta + p^T \gamma - E\alpha,\ \ell, g, p \in \mathbb{R}^3 \ \wedge\ E \in \mathbb{R} \right\}. \tag{49}$$

We have the following definitions.

Definition 1.22

Let
$$\mu := \{\ M \mid P\ \} := \{\ \ell \mid g \mid p \mid E\ \} \in \mathfrak{poin}^*$$
with the relations:
$$M = \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \quad , \quad P = \begin{pmatrix} E \\ p \end{pmatrix}.$$

(i) (a) The vector $\ell(\mu) := \ell \in \mathbb{R}^3$ is called the **angular momentum of** M **associated with** μ.

(b) The vector $g(\mu) := g \in \mathbb{R}^3$ is the **relativistic barycenter of** M **associated with** μ.

(ii) (a) The vector $p(\mu) := p \in \mathbb{R}^3$ is called the **linear momentum of** P **associated with** μ.

(b) The scalar $E(\mu) := E \in \mathbb{R}$ is called the **energy of** P **associated with** μ.

We deduce a simple expression for the action of the **PT**-group $\mathscr{K}$ on the torsors of $\mathfrak{poin}^*$.

Proposition 1.23

Let $\{\ \ell \mid g \mid p \mid E\ \} \in \mathfrak{poin}^*$ and $\lambda, \nu \in \{0,1\}$. We have:
$$(\mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \{\ \ell \mid g \mid p \mid E\ \} = \{\ l \mid (-1)^{\lambda+\nu} g \mid (-1)^\nu p \mid (-1)^\lambda E\ \}$$

Proof. Let $\mu := \{\ M \mid P\ \} := \{\ \ell \mid g \mid p \mid E\ \} \in \mathfrak{poin}^*$. As for any $\mathbf{A} := \begin{pmatrix} \mathbf{L} & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin$:
$$\begin{aligned} M(\mathbf{A} \bullet \mu) &= \mathbf{L} M(\mu) \tau(\mathbf{L}) + C \tau(P(\mu)) \tau(\mathbf{L}) + \mathbf{L} P(\mu) \tau(C) \\ P(\mathbf{A} \bullet \mu) &= \mathbf{L} P(\mu) \end{aligned}$$
we have:
$$\begin{aligned} (\mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \{\ \ell \mid g \mid p \mid E\ \} &= (\mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \left\{\ \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \middle| \begin{pmatrix} E \\ p \end{pmatrix}\ \right\} \\ &= \left\{\ \widetilde{\mathbf{P}}^\nu \widetilde{\mathbf{T}}^\lambda \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} \widetilde{\mathbf{T}}^\lambda \widetilde{\mathbf{P}}^\nu \ \middle|\ I_{1,3} \widetilde{\mathbf{T}}^\lambda \widetilde{\mathbf{P}}^\nu I_{1,3} \begin{pmatrix} E \\ p \end{pmatrix}\ \right\} \\ &= \left\{\ \begin{pmatrix} 0 & (-1)^{\lambda+\nu} g^T \\ (-1)^{\lambda+\nu} g & j(\ell) \end{pmatrix} \middle| \begin{pmatrix} (-1)^\lambda E \\ (-1)^\nu p \end{pmatrix}\ \right\} \\ &= \{\ l \mid (-1)^{\lambda+\nu} g \mid (-1)^\nu p \mid (-1)^\lambda E\ \} \end{aligned}$$
□

For any $\{\ \ell \mid g \mid p \mid E\ \} \in \mathfrak{poin}^*$, we have:
$$\mathbf{P} \bullet \{\ \ell \mid g \mid p \mid E\ \} = \{\ \ell \mid -g \mid -p \mid E\ \} \tag{50}$$
$$\mathbf{T} \bullet \{\ \ell \mid g \mid p \mid E\ \} = \{\ \ell \mid -g \mid p \mid -E\ \} \tag{51}$$

1.8 Stabilizer of a Point

Definition 1.24

Let $X \in \mathbb{R}^{1,3}$ and $\mathbf{B} \in \mathscr{G}$. The **stabilizer of** X **by** $\mathscr{P}oin$ is defined as:
$$\mathscr{P}oin_X := \{\mathbf{A} \in \mathscr{P}oin,\ \ \mathbf{A} \bullet X = X\}$$

We have simple equivalences:

$$\mathbf{A} := \begin{pmatrix} \mathbf{L} & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin_X \iff \mathbf{A} \bullet X = X \iff \mathbf{L}X + C = X \iff C = (I_4 - \mathbf{L})X$$

We thus have the representation:

$$\mathscr{P}oin_X = \left\{ \begin{pmatrix} \mathbf{L} & (I_4 - \mathbf{L})X \\ 0 & 1 \end{pmatrix}, \ \mathbf{L} \in \mathscr{L}or \right\}. \tag{52}$$

From this, we derive the following result.

Lemma 1.25

Let $X \in \mathbb{R}^{1,3}$. The group $\mathscr{P}oin_X$ is a Lie subgroup of $\mathscr{P}oin$ with dimension 6, and its Lie algebra is the Lie subalgebra of $\mathfrak{poin}$ with dimension 6 defined by:

$$\mathfrak{poin}_X = \left\{ \begin{pmatrix} \Lambda & -\Lambda X \\ 0 & 0 \end{pmatrix}, \ \Lambda \in \mathscr{A}(1,3,\mathbb{R}) \right\}.$$

Proof. From equation (16), and since $\mathfrak{poin}_X$ is a Lie subalgebra of $\mathfrak{poin}$, the Lie algebra $\mathfrak{poin}_X$ of $\mathscr{P}oin_X$ is given by:

$$\mathfrak{poin}_X := \mathrm{T}_{I_4}\mathscr{P}oin_X = \left\{ M \in \mathfrak{poin}, \ \forall t \in \mathbb{R}, \ e^{tM} \in \mathscr{P}oin_X \right\}.$$

Let $\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \in \mathfrak{poin}$. We have two cases.

(1) Case $\Lambda = 0$. Then we have:

$$\begin{aligned}
\begin{pmatrix} 0 & \Gamma \\ 0 & 0 \end{pmatrix} \in \mathfrak{poin}_X &\iff \forall t \in \mathbb{R}, \ \exists \mathbf{L}_t \in \mathscr{L}or, \ \exp\left(t \begin{pmatrix} 0 & \Gamma \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} \mathbf{L}_t & (I_4 - \mathbf{L}_t)X \\ 0 & 1 \end{pmatrix} \\
&\iff \forall t \in \mathbb{R}, \ \exists \mathbf{L}_t \in \mathscr{L}or, \ \begin{pmatrix} I_4 & t\Gamma \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{L}_t & (I_4 - \mathbf{L}_t)X \\ 0 & 1 \end{pmatrix} \\
&\iff \forall t \in \mathbb{R}, \ t\Gamma = 0 \\
&\iff \Gamma = 0 = -\Lambda X
\end{aligned}$$

(2) Case $\Lambda \neq 0$. Since $e^{t\Lambda} \in \mathscr{L}or$ for all $t \in \mathbb{R}$, we have:

$$\begin{aligned}
&\begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \in \mathfrak{poin}_X \\
\iff &\forall t \in \mathbb{R}, \ \exists \mathbf{L}_t \in \mathscr{L}or, \ \exp\left(t \begin{pmatrix} \Lambda & \Gamma \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} \mathbf{L}_t & (I_4 - \mathbf{L}_t)X \\ 0 & 1 \end{pmatrix} \\
\iff &\forall t \in \mathbb{R}, \ \exists \mathbf{L}_t \in \mathscr{L}or, \ \begin{pmatrix} e^{t\Lambda} & U \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{L}_t & (I_4 - \mathbf{L}_t)X \\ 0 & 1 \end{pmatrix} \ \wedge \ \Lambda U = (e^{t\Lambda} - I_4)\Gamma \\
\iff &\forall t \in \mathbb{R}, \ (e^{t\Lambda} - I_4)\Gamma = (I_4 - e^{t\Lambda})\Lambda X \\
\iff &\forall t \in \mathbb{R}, \ \Gamma + \Lambda X = e^{t\Lambda}(\Gamma + \Lambda X) \\
\iff &\Gamma = -\Lambda X
\end{aligned}$$

□

Thus, for $X := \begin{pmatrix} t \\ r \end{pmatrix} \in \mathbb{R}^{1,3}$, we have:

$$\mathscr{B}ase(\mathfrak{poin}_X) = \left\{ \begin{pmatrix} e_i \ominus 0 & -(e_i \ominus 0)X \\ 0 & 0 \end{pmatrix}, \ i \in \{1,2,3\} \right\} \bigsqcup \left\{ \begin{pmatrix} 0 \ominus e_i & -(0 \ominus e_i)X \\ 0 & 0 \end{pmatrix}, \ i \in \{1,2,3\} \right\}$$

$$= \left\{ \begin{pmatrix} 0 & e_i^T & -e_i^T r \\ e_i & 0 & -te_i \\ 0 & 0 & 0 \end{pmatrix},\ i \in \{1,2,3\} \right\} \bigsqcup \left\{ \begin{pmatrix} 0 & 0 & e_i^T r \\ 0 & j(e_i) & r \wedge e_i \\ 0 & 0 & 0 \end{pmatrix},\ i \in \{1,2,3\} \right\}.$$

Definition 1.26

Let $\mu \in \mathfrak{poin}$ and $X \in \mathbb{R}^{1,3}$. The **tensor with respect to** X **associated with** μ is defined by the restriction of μ to the Lie subalgebra $\mathfrak{poin}_X$, i.e.:

$$\mu_X := \mu_{|\mathfrak{poin}_X} : \begin{array}{ccc} \mathfrak{poin}_X & \longrightarrow & \mathbb{R} \\ \begin{pmatrix} \Lambda & -\Lambda X \\ 0 & 0 \end{pmatrix} & \longmapsto & \mu\left(\begin{pmatrix} \Lambda & -\Lambda X \\ 0 & 0 \end{pmatrix}\right) \end{array}$$

Let us define, for every $X \in \mathbb{R}^{1,3}$ and every $M \in \mathscr{A}(1,3,\mathbb{R})$:

$$M_X := M + P\tau(X) - X\tau(P) \in \mathscr{A}(1,3,\mathbb{R}). \tag{53}$$

We then have the following characterization of $\mathfrak{poin}_X^*$.

Proposition 1.27

Let $X \in \mathbb{R}^{1,3}$. Then we have the following characterization:

$$\mathfrak{poin}_X^* = \left\{ \left\{\ M_X\ \right\} : \begin{pmatrix} \Lambda & -\Lambda X \\ 0 & 0 \end{pmatrix} \longmapsto -\frac{1}{2}\mathrm{Tr}(M_X \Lambda),\ M \in \mathscr{A}(1,3,\mathbb{R}) \right\}.$$

Proof. Let $\mu := \left\{\ M \mid P\ \right\} \in \mathfrak{poin}^*$ and $\begin{pmatrix} \Lambda & -\Lambda X \\ 0 & 0 \end{pmatrix} \in \mathfrak{poin}$. We have:

$$\begin{aligned} \mu_X \begin{pmatrix} \Lambda & -\Lambda X \\ 0 & 0 \end{pmatrix} = \mu \begin{pmatrix} \Lambda & -\Lambda X \\ 0 & 0 \end{pmatrix} &= \left\{\ M \mid P\ \right\} \begin{pmatrix} \Lambda & -\Lambda X \\ 0 & 0 \end{pmatrix} \\ &= -\frac{1}{2}\mathrm{Tr}(M\Lambda) + \tau(P)\Lambda X \\ &= -\frac{1}{2}\left(\mathrm{Tr}(M\Lambda) - 2\tau(P)\Lambda X\right) \\ &= -\frac{1}{2}\left(\mathrm{Tr}(M\Lambda) - \mathrm{Tr}(\tau(P)\Lambda X) - \mathrm{Tr}(\tau(\tau(P)\Lambda X))\right) \\ &= -\frac{1}{2}\left(\mathrm{Tr}(M\Lambda) - \mathrm{Tr}(X\tau(P)\Lambda) + \mathrm{Tr}(P\tau(X)\Lambda)\right) \\ &= -\frac{1}{2}\mathrm{Tr}((M + P\tau(X) - X\tau(P))\Lambda) \\ &= -\frac{1}{2}\mathrm{Tr}(M_X\Lambda) \end{aligned}$$

□

Thus, we have a natural surjective linear map:

$$\begin{array}{ccc} \mathfrak{poin}^* & \longrightarrow & \mathfrak{poin}_X^* \\ \mu := \left\{\ M \mid P\ \right\} & \longmapsto & \mu_X := \left\{\ M_X\ \right\} \end{array} \tag{54}$$

From this, we derive the following definitions.

Definition 1.28

Let $\mu := \left\{ \; M \; \middle| \; P \; \right\} \in \mathfrak{poin}$ and $X \in \mathbb{R}^{1,3}$. The **Lorentz moment with respect to** X **associated with** μ is defined as:

$$M(\mu)_X := M_X := \begin{pmatrix} 0 & g_X^T \\ g_X & j(\ell_X) \end{pmatrix} := M + P\tau(X) - X\tau(P)$$

(i) We call $g(\mu)_X := g_X$ the **centroid with respect to** X **associated with** μ.

(ii) We call $\ell(\mu)_X := \ell_X$ the **angular momentum with respect to** X **associated with** μ.

Let $X := \begin{pmatrix} t \\ r \end{pmatrix} \in \mathbb{R}^{1,3}$. We directly have:

$$\begin{aligned}
\begin{pmatrix} 0 & g_X^T \\ g_X & j(\ell_X) \end{pmatrix} &= M + P\tau(X) - X\tau(P) \\
&= \begin{pmatrix} 0 & g^T \\ g & j(\ell) \end{pmatrix} + \begin{pmatrix} Et & -Er^T \\ tp & -pr^T \end{pmatrix} - \begin{pmatrix} Et & tp^T \\ -Er & -rp^T \end{pmatrix} \\
&= \begin{pmatrix} 0 & g^T - Er^T + tp^T \\ g + tp - Er & j(\ell + p \wedge r) \end{pmatrix}
\end{aligned}$$

i.e. we have:

$$g_X = g + tp - Er \tag{55}$$

$$\ell_X = \ell + p \wedge r \tag{56}$$

We thus deduce the following simple formula.

Proposition 1.29

Let $\mathbf{A} := \begin{pmatrix} \mathbf{L} & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin$ and $X \in \mathbb{R}^{1,3}$. We have:

$$M\left(\mathbf{A} \bullet \mu\right)_{\mathbf{A} \bullet X} = \mathbf{L} M(\mu)_X \tau(\mathbf{L}).$$

Proof. Let us define:

$$X' := \mathbf{A} \bullet X = \mathbf{L}X + C \quad , \quad \mu' := \left\{ \; M' \; \middle| \; P' \; \right\} := \mathbf{A} \bullet \mu$$

We have:

$$\begin{aligned}
M'_{X'} &= M' + P'\tau(X') - X'\tau(P') \\
&= \mathbf{L}M\tau(\mathbf{L}) + C\tau(P)\tau(\mathbf{L}) - \mathbf{L}P\tau(C) + \mathbf{L}P\tau(\mathbf{L}X + C) - (\mathbf{L}X + C)\tau(\mathbf{L}P) \\
&= \mathbf{L}(M + P\tau(X) - X\tau(P))\tau(\mathbf{L}) \\
&= \mathbf{L}M_X\tau(\mathbf{L})
\end{aligned}$$

□

1.9 Polarization and Casimir Numbers

In this subsection, we define polarization, also called the **Pauli-Lubanski pseudovector** (see the appendix for the definition of the Hodge operator), and the two Casimir numbers. Let us start with a lemma that justifies the definition of polarization.

Lemma 1.30

Let $\mu := \{ \; M \mid P \; \} \in \mathfrak{poin}^*$. Then the mapping:

$$X \in \mathbb{R}^{1,3} \longmapsto *(M_X)P$$

is constant at $*(M)P$.

Proof. This follows from point (iii) of Proposition C.2. □

This justifies the following definition.

Definition 1.31

Let $\mu := \{ \; M \mid P \; \} \in \mathfrak{poin}^*$. The **polarization vector associated with** μ is defined by:

$$W(\mu) := W := *(M)P$$

Let us denote:

$$\mu := \{ \; M \mid P \; \} := \{ \; \ell \mid g \mid p \mid E \; \} \;, \quad M := \begin{pmatrix} 0 & g^T \\ g & j(l) \end{pmatrix} \;, \quad P := \begin{pmatrix} E \\ p \end{pmatrix}$$

Then by the previous lemma:

$$W = *(M)P = \begin{pmatrix} 0 & \ell^T \\ \ell & j(-g) \end{pmatrix} \begin{pmatrix} E \\ p \end{pmatrix} = \begin{pmatrix} \ell^T p \\ j(-g)p + \ell E \end{pmatrix} = \begin{pmatrix} \ell^T p \\ p \wedge g + \ell E \end{pmatrix}. \tag{57}$$

By point (i) of proposition C.2, the vectors P and W are orthogonal in $\mathbb{R}^{1,3}$, i.e.

$$\tau(P)W = 0. \tag{58}$$

The action of the Poincaré group on the polarization is given by the following formula.

Proposition 1.32

Let:

$$\mu \in \mathfrak{poin}^* \quad , \quad \mathbf{A} := \begin{pmatrix} \mathbf{L} & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin.$$

We have:

$$W(\mathbf{A} \bullet \mu) = \det(\mathbf{L})\mathbf{L}W(\mu).$$

Proof. Let us denote:

$$P := P(\mu) \qquad M := M(\mu) \qquad W := W(\mu)$$
$$P' := P(\mathbf{A} \bullet \mu) \qquad M' := M(\mathbf{A} \bullet \mu) \qquad W' := W(\mathbf{A} \bullet \mu)$$

By corollary 1.21, we have:

$$M' = \mathbf{L}M\tau(\mathbf{L}) + C\tau(\mathbf{L}P) + \mathbf{L}P\tau(C) \qquad\qquad P' = \mathbf{L}P$$

Thus, by the proposition C.5 from the appendix, we have:

$$\begin{aligned} W' &= *(M')P' \\ &= *(\mathbf{L}M\tau(\mathbf{L}) + C\tau(\mathbf{L}P) + \mathbf{L}P\tau(C))\mathbf{L}P \end{aligned}$$

$$= *(\mathbf{L}M\tau(\mathbf{L}))\mathbf{L}P + *(C\tau(\mathbf{L}P) + \mathbf{L}P\tau(C))\mathbf{L}P$$
$$= \det(\mathbf{L})\mathbf{L} * (M)\tau(\mathbf{L})\mathbf{L}P + j_4(C, \mathbf{L}P)\mathbf{L}P$$
$$= \det(\mathbf{L})\mathbf{L}W$$

□

Thus we have in particular:

$$W(\mathbf{P} \bullet \mu) = -\mathbf{P}W(\mu) = \mathbf{T}W(\mu) \tag{59}$$
$$W(\mathbf{T} \bullet \mu) = -\mathbf{T}W(\mu) = \mathbf{P}W(\mu) \tag{60}$$

We then deduce two constants invariant under the Poincaré group, which will form the basis for two other numbers associated with μ: mass and spin.

Definition 1.33

Let $\mu := \{\ M \mid P\ \} \in \mathfrak{poin}^*$.

(i) The **second Casimir number associated with** μ is defined by:

$$C_2 := C_2(\mu) := \tau(P)P.$$

(ii) The **fourth Casimir number associated with** μ is defined by:

$$C_4 := C_4(\mu) := \tau(W)W.$$

For:

$$\mu := \{\ M \mid P\ \} := \{\ \ell \mid g \mid p \mid E\ \} \in \mathfrak{poin}^*$$

we have:

$$C_2 = \tau(P)P = \begin{pmatrix} E & -p^T \end{pmatrix} \begin{pmatrix} E \\ p \end{pmatrix} = E^2 - p^T p. \tag{61}$$

As stated before the definition, let us show the invariance of these two numbers under the Poincaré group.

Lemma 1.34

Let $\mu \in \mathfrak{poin}^*$ and $\mathbf{A} \in \mathscr{P}oin$. We have:

$$C_2(\mathbf{A} \bullet \mu) = C_2(\mu)$$
$$C_4(\mathbf{A} \bullet \mu) = C_4(\mu)$$

Proof. Let us denote:

$$\mu := \{\ M \mid P\ \} \quad , \quad \mathbf{A} := \begin{pmatrix} \mathbf{L} & C \\ 0 & 1 \end{pmatrix}.$$

and:

$$P := P(\mu) \qquad M := M(\mu) \qquad W := W(\mu)$$
$$P' := P(\mathbf{A} \bullet \mu) \qquad M' := M(\mathbf{A} \bullet \mu) \qquad W' := W(\mathbf{A} \bullet \mu)$$

We have:

$$C_2(\mathbf{A} \bullet \mu) = \tau(P')P' = \tau(\mathbf{L}P)\mathbf{L}P = \tau(P)\tau(\mathbf{L})\mathbf{L}P = C_2(\mu)$$
$$C_4(\mathbf{A} \bullet \mu) = \tau(W')W' = \tau(\det(\mathbf{L})\mathbf{L}W)\det(\mathbf{L})\mathbf{L}W = \tau(W)\tau(\mathbf{L})\mathbf{L}W = C_4(\mu)$$

□

2 Motion of Matter in Minkowski Space

2.1 Real Matter

We start by defining the notion of real matter.[2]

Definition 2.1

Let $\mu \in \mathfrak{poin}^*$. We say that μ is **(real) matter** if:

$$C_2(\mu) \geq 0.$$

We denote by $\mathscr{M}at$ the set of (real) matter, i.e.,

$$\mathscr{M}at := \{\mu \in \mathfrak{poin}^*,\ C_2(\mu) \geq 0\}.$$

We then deduce the definition of the mass of a matter.

Definition 2.2

Let $\mu \in \mathscr{M}at$.

(i) The **sign of E associated with** μ is defined by:

$$\zeta(\mu) := \zeta := \operatorname{sign}(E(\mu)) \in \{\pm 1\}.$$

(ii) The **mass associated with** μ is defined by[a]:

$$m(\mu) := m := \zeta(\mu)\sqrt{C_2(\mu)}.$$

[a] Since μ is real matter, we have $C_2 \geq 0$, and thus the square root exists.

For a matter:

$$\mu := \left\{\ M \mid P\ \right\} := \left\{\ \ell \mid g \mid p \mid E\ \right\} \in \mathscr{M}at$$

we have:

$$m(\mu) = \zeta(\mu)\sqrt{E^2 - p^T p}.$$

Note that no constraint is given on the sign of the mass; it can be negative or positive.

Proposition 2.3

Let

$$\mathbf{A} := \begin{pmatrix} \mathbf{L}_n \widetilde{\mathbf{P}}^{\nu} \widetilde{\mathbf{T}}^{\lambda} & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin \quad , \quad \mu \in \mathfrak{poin}^*.$$

We have:

$$\zeta\left(\mathbf{A} \bullet \mu\right) = (-1)^{\lambda} \zeta\left(\mu\right) \quad , \quad m\left(\mathbf{A} \bullet \mu\right) = (-1)^{\lambda} m(\mu).$$

Proof. Let us denote:

$$\mu := \left\{\ M \mid P\ \right\} := \left\{\ \ell \mid g \mid p \mid E\ \right\} \in \mathscr{M}at.$$

We will handle two cases separately:

$$\mathbf{A} \in \mathscr{K} \quad , \quad \mathbf{A} \in \mathscr{P}oin_n.$$

[2] The notion of pure imaginary matter can be defined by the condition $C_2 < 0$.

(1) Case $\mathbf{A} := \mathbf{P}^\nu\mathbf{T}^\lambda$. We have by proposition 1.23:

$$\zeta\left(\left(\mathbf{P}^\nu\mathbf{T}^\lambda\right)\bullet\mu\right) = \operatorname{sign}\left(E\left(\left(\mathbf{P}^\nu\mathbf{T}^\lambda\right)\bullet\mu\right)\right) = \operatorname{sign}((-1)^\lambda E) = (-1)^\lambda\zeta(\mu)$$

and thus we have:

$$m\left(\left(\mathbf{P}^\nu\mathbf{T}^\lambda\right)\bullet\mu\right) = \zeta\left(\left(\mathbf{P}^\nu\mathbf{T}^\lambda\right)\bullet\mu\right)\sqrt{C_2\left(\left(\mathbf{P}^\nu\mathbf{T}^\lambda\right)\bullet\mu\right)} = (-1)^\lambda\zeta(\mu)\sqrt{C_2(\mu)} = (-1)^\lambda m(\mu)$$

(2) Case:

$$\mathbf{A} := \begin{pmatrix}\mathbf{L}_n & C\\ 0 & 1\end{pmatrix} \in \mathscr{P}oin_n \quad , \quad \mathbf{L}_n := \begin{pmatrix} a & b^T\\ c & d\end{pmatrix} \in \mathscr{L}or_n.$$

By definition, we have $a = [\mathbf{L}_n]_{0,0} \geq 1$. Since $\mathbf{L}_n^T I_{1,3}\mathbf{L}_n = I_{1,3}$, we have:

$$1 = \eta_{0,0} = \left[\mathbf{L}_n^T\right]_{0,i}\eta_{i,i}\left[\mathbf{L}_n\right]_{i,0} = a^2 - b^T b$$

And since:

$$\begin{pmatrix}E(\mathbf{A}\bullet\mu)\\ p(\mathbf{A}\bullet\mu)\end{pmatrix} = P(\mathbf{A}\bullet\mu) = \mathbf{L}_n P(\mu) = \begin{pmatrix} a & b^T\\ c & d\end{pmatrix}\begin{pmatrix}E\\ p\end{pmatrix} = \begin{pmatrix}aE + b^T p\\ Ec + dp\end{pmatrix} \tag{62}$$

we have $E(\mathbf{A}\bullet\mu) = aE + b^T p$. Thus, since $E^2 - p^T p = C_2 \geq 0$ (i.e., $E^2 \geq p^T p$), we apply the Cauchy-Schwarz inequality in $\mathbb{R}^3$:

$$|b^T p|^2 \leq (b^T b)(p^T p) = (a^2 - 1)(p^T p) < (aE)^2$$

Thus we have:

$$\zeta(\mathbf{A}\bullet\mu) = \operatorname{sign}\left(E(\mathbf{A}\bullet\mu)\right) = \operatorname{sign}\left(aE\right) = \operatorname{sign}\left(E\right) = \zeta(\mu).$$

From which we have:

$$m\left(\mathbf{A}\bullet\mu\right) = \zeta\left(\mathbf{A}\bullet\mu\right)\sqrt{C_2\left(\mathbf{A}\bullet\mu\right)} = \zeta(\mu)\sqrt{C_2(\mu)} = m(\mu)$$

□

Thus the sign of energy and mass are invariant under the orthochronous subgroup of the Poincaré group $\mathscr{P}oin_o$, and we have for the time-reversed part:

$$\zeta\left(\mathbf{T}\bullet\mu\right) = -\zeta(\mu)$$
$$m\left(\mathbf{T}\bullet\mu\right) = -m(\mu).$$

2.2 Type of Matter

We recall the notion of the type of a vector in Minkowski space.

Definition 2.4

Let $X := \begin{pmatrix} t\\ r\end{pmatrix} \in \mathbb{R}^{1,3}$.

(i) We say that X is of **time type** if $\tau(X)X > 0$, i.e., $|t| > \|r\|$.

(ii) We say that X is of **space type** if $\tau(X)X < 0$, i.e., $|t| < \|r\|$.

(iii) We say that X is of **isotropic type** if $X \neq 0$ and if $\tau(X)X = 0$, i.e., $|t| = \|r\|$.

We will classify matter according to the types of the vectors $P(\mu)$ and $W(\mu)$ associated with a matter μ, i.e., according to the signs of C_2 and C_4. Let us begin with a lemma that justifies the study of only these three cases.

Lemma 2.5

Let $\mu \in \mathscr{M}at$ such that $C_2(\mu) > 0$. Then we have $C_4(\mu) \leq 0$.

Proof. Suppose for contradiction that $C_4(\mu) > 0$, i.e., that $P(\mu)$ and $W(\mu)$ are of time type. Then by the reversed Cauchy-Schwarz inequality in Minkowski space for time-type vectors, we have:

$$0 = \tau(P(\mu))W(\mu) \geq C_2(\mu)C_4(\mu) > 0$$

which is absurd. Thus, $C_4(\mu) \leq 0$. □

We then study the following two cases.

Case I. $C_2(\mu), C_4(\mu) > 0$: real matter of non-zero mass with spin. We denote their set by:

$$\mathscr{M}at(\mathbf{I}) := \{\mu \in \mathscr{M}at,\ C_2(\mu) > 0 \ \wedge\ C_4(\mu) > 0\}\,. \tag{63}$$

Case II. $C_2(\mu) > 0$ and $C_4(\mu) = 0$: real matter of non-zero mass without spin. We denote their set by:

$$\mathscr{M}at(\mathbf{II}) := \{\mu \in \mathscr{M}at,\ C_2(\mu) > 0 \ \wedge\ C_4(\mu) = 0\}\,. \tag{64}$$

We will use the following notation by abuse:

$$\mathscr{M}at(\mathbf{I} \cup \mathbf{II}) := \mathscr{M}at(\mathbf{I}) \cup \mathscr{M}at(\mathbf{II}) = \{\mu \in \mathscr{M}at,\ C_2(\mu) > 0 \ \wedge\ C_4(\mu) \geq 0\}\,. \tag{65}$$

Jean-Marie Souriau studies another case that we will not address in this text:

Case III. $C_2(\mu), C_4(\mu) = 0$ and $P(\mu), W(\mu) \neq 0$: real matter without mass and with spin. We denote their set by:

$$\mathscr{M}at(\mathbf{III}) := \{\mu \in \mathscr{M}at,\ C_2(\mu) = 0 \ \wedge\ C_4(\mu) = 0 \ \wedge\ P(\mu),\ W(\mu) \neq 0\}\,. \tag{66}$$

2.3 Trajectory Associated with Matter of Non-Zero Mass

For each case, we will study the possible trajectories of real matter[3].

Definition 2.6

Let $\mu \in \mathscr{M}at(\mathbf{I} \cup \mathbf{II})$. The **universal trajectory associated with** μ is defined by:

$$\mathfrak{T}(\mu) := \left\{X \in \mathbb{R}^{1,3},\ M(\mu)_X P(\mu) = 0\right\}.$$

For any subset $\mathscr{E}$ of $\mathbb{R}^{1,3}$ and any $\mathbf{A} := \begin{pmatrix} \mathbf{L} & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin$, we set:

$$\mathbf{A} \bullet \mathscr{E} := \{\mathbf{A} \bullet X,\ X \in \mathscr{E}\} = \{\mathbf{L}X + C,\ X \in \mathscr{E}\}\,. \tag{67}$$

Proposition 2.7

Let:

$$\mathbf{A} := \begin{pmatrix} \mathbf{L} & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin\ ,\quad \mu \in \mathscr{M}at(\mathbf{I} \cup \mathbf{II}).$$

We have:

$$\mathfrak{T}(\mathbf{A} \bullet \mu) = \mathbf{A} \bullet \mathfrak{T}(\mu).$$

[3] The following definitions also extend to any torsor.

Proof. Let:

$$\mu := \left\{ \begin{array}{c|c} M & P \end{array} \right\} \quad , \quad \mu' := \left\{ \begin{array}{c|c} M' & P' \end{array} \right\} := \mathbf{A} \bullet \mu.$$

By Proposition 1.29, we have:

$$M'_{\mathbf{A}\bullet X} P' = \mathbf{L} M_X \tau(\mathbf{L}) \mathbf{L} P = \mathbf{L} M_X P.$$

Thus, we have:

$$\begin{aligned} \mathfrak{T}(\mu') &= \left\{ X \in \mathbb{R}^{1,3},\ M'_X P' = 0 \right\} \\ &= \mathbf{A} \bullet \left\{ X \in \mathbb{R}^{1,3},\ M'_{\mathbf{A}\bullet X} P' = 0 \right\} \\ &= \mathbf{A} \bullet \left\{ X \in \mathbb{R}^{1,3},\ \mathbf{L} M_X P = 0 \right\} \\ &= \mathbf{A} \bullet \left\{ X \in \mathbb{R}^{1,3},\ M_X P = 0 \right\} \\ &= \mathbf{A} \bullet \mathfrak{T}(\mu) \end{aligned}$$

□

In particular, we have for all $\mu \in \mathscr{M}at(\mathbf{I} \cup \mathbf{II})$ and all $\lambda, \nu \in \{0, 1\}$:

$$\mathfrak{T}\left((\mathbf{P}^\nu \mathbf{T}^\lambda) \bullet \mu \right) = \mathbf{P}^\nu \mathbf{T}^\lambda \bullet \mathfrak{T}(\mu) = \left\{ \begin{pmatrix} (-1)^\lambda t \\ (-1)^\nu r \end{pmatrix},\ \begin{pmatrix} t \\ r \end{pmatrix} \in \mathfrak{T}(\mu) \right\}. \tag{68}$$

We have a natural mapping:

$$\begin{array}{rccc} \mathfrak{T}: & \mathscr{M}at(\mathbf{I} \cup \mathbf{II}) & \longrightarrow & \mathscr{P}(\mathbb{R}^{1,3}) \\ & \mu & \longmapsto & \mathfrak{T}(\mu) \end{array} \tag{69}$$

The image of this mapping is the set of all possible trajectories associated with torsors of $\mathfrak{poin}$ in $\mathbb{R}^{1,3}$.

Definition 2.8

The **set of trajectories in** $\mathbb{R}^{1,3}$ is defined by:

$$\mathbb{T} := \mathrm{Im}(\mathfrak{T}) := \{\mathfrak{T}(\mu),\ \mu \in \mathscr{M}at(\mathbf{I} \cup \mathbf{II})\}.$$

2.4 Case of Real Matter with Non-Zero Mass

In this subsection, we study real matter with non-zero mass. Let's start with a simple lemma that justifies dividing by m and E.

Lemma 2.9

Let $\mu \in \mathscr{M}at(\mathbf{I} \cup \mathbf{II})$. Then $m(\mu)$ and $E(\mu)$ are non-zero.

Proof. We have:

$$m(\mu)^2 = E(\mu)^2 - p(\mu)^T p(\mu) = C_2(\mu) > 0$$

thus $E(\mu)^2, m(\mu)^2 > 0$ *i.e.* $E(\mu), m(\mu) \neq 0$. □

Definition 2.10

Let $\mu \in \mathscr{M}at(\mathbf{I} \cup \mathbf{II})$.

(i) The **spin associated with** μ is defined by:

$$s(\mu) := s := \sqrt{\frac{-C_4(\mu)}{C_2(\mu)}}.$$

(ii) The **unit energy-momentum four-vector associated with** μ is the vector in $\mathbb{R}^{1,3}$ defined by:

$$I(\mu) := I := \frac{1}{m(\mu)} P(\mu)$$

(iii) The **velocity vector associated with** μ is the vector in $\mathbb{R}^3$ defined by:

$$v(\mu) := v := \frac{1}{E(\mu)} p(\mu).$$

Thus, for every $\mu \in \mathscr{M}at(\mathbf{I} \cup \mathbf{II})$:

$$s(\mu) \neq 0 \iff \mu \in \mathscr{M}at(\mathbf{I}). \tag{70}$$

We can thus define an additional vector in the case of matter from $\mathscr{M}at(\mathbf{I})$.

Definition 2.11: Polarization

Let $\mu \in \mathscr{M}at(\mathbf{I})$. The **unit polarization vector associated with** μ is the vector in $\mathbb{R}^{1,3}$ defined by:

$$J(\mu) := J := \frac{1}{s(\mu)m(\mu)} W(\mu).$$

From this, we deduce the action of the Poincaré group on these elements.

Proposition 2.12

Let

$$\mathbf{A} := \begin{pmatrix} \mathbf{L}_n \widetilde{\mathbf{P}}^{\nu} \widetilde{\mathbf{T}}^{\lambda} & C \\ 0 & 1 \end{pmatrix} \in \mathscr{P}oin.$$

(i) For every $\mu \in \mathscr{M}at(\mathbf{I} \cup \mathbf{II})$, we have:

$$s(\mathbf{A} \bullet \mu) = s(\mu)$$
$$I(\mathbf{A} \bullet \mu) = \mathbf{L}_n \widetilde{\mathbf{P}}^{\lambda+\nu} I(\mu)$$

(ii) For every $\mu \in \mathscr{M}at(\mathbf{I})$, we have:

$$J(\mathbf{A} \bullet \mu) = \mathbf{L}_n \widetilde{\mathbf{T}}^{\lambda+\nu} J(\mu).$$

Proof. By the invariance of the Casimir numbers under the Poincaré group (see lemma 1.34), we have:

$$s(\mathbf{A} \bullet \mu) = \sqrt{\frac{-C_4(\mathbf{A} \bullet \mu)}{C_2(\mathbf{A} \bullet \mu)}} = \sqrt{\frac{-C_4(\mu)}{C_2(\mu)}} = s(\mu).$$

We have with $\mathbf{L} := \mathbf{L}_n \widetilde{\mathbf{P}}^{\nu} \widetilde{\mathbf{P}}^{\lambda}$:

$$I(\mathbf{A} \bullet \mu) = \frac{P(\mathbf{A} \bullet \mu)}{m(\mathbf{A} \bullet \mu)} = \frac{\mathbf{L}P(\mu)}{(-1)^{\lambda} m(\mu)} = (-1)^{\lambda} \mathbf{L} I(\mu) = \mathbf{L}_n \widetilde{\mathbf{P}}^{\lambda+\nu} I(\mu)$$

$$J(\mathbf{A}\bullet\mu) = \frac{W(\mathbf{A}\bullet\mu)}{s(\mathbf{A}\bullet\mu)\,m(\mathbf{A}\bullet\mu)} = \frac{\det(\mathbf{L})\mathbf{L}W(\mu)}{s(\mu)(-1)^{\lambda}m(\mu)} = \mathbf{L}_n\widetilde{\mathbf{T}}^{\lambda+\nu}J(\mu)$$

□

Thus, we have:

$$I(\mathbf{T}\bullet\mu) = \mathbf{P}I(\mu) \qquad J(\mathbf{T}\bullet\mu) = \mathbf{T}J(\mu) \tag{71}$$

$$I(\mathbf{P}\bullet\mu) = \mathbf{P}I(\mu) \qquad J(\mathbf{P}\bullet\mu) = \mathbf{T}J(\mu) \tag{72}$$

There is no simple formula for $v(\mathbf{A}\bullet\mu)$ in terms of $v(\mu)$; we only have:

$$v\left(\left(\mathbf{P}^{\nu}\mathbf{T}^{\lambda}\right)\bullet\mu\right) = \frac{p\left(\left(\mathbf{P}^{\nu}\mathbf{T}^{\lambda}\right)\bullet\mu\right)}{E\left(\left(\mathbf{P}^{\nu}\mathbf{T}^{\lambda}\right)\bullet\mu\right)} = \frac{(-1)^{\nu}p(\mu)}{(-1)^{\lambda}E(\mu)} = (-1)^{\lambda+\nu}v(\nu) \tag{73}$$

We also have the usual formula:

$$E = \frac{\operatorname{sign}(E)^2E\sqrt{C_2}}{\sqrt{E-p^Tp}} = \frac{m}{\sqrt{1-v^Tv}} \tag{74}$$

Theorem 2.13

Let $\mu \in \mathscr{M}at(\mathbf{I}\cup\mathbf{II})$. Let's set:

(i) (a) We have $\tau(I)I = 1$.

(b) Suppose that $\mu \in \mathscr{M}at(\mathbf{I})$. We have:

$$\tau(I)I = 1 \quad , \quad \tau(J)J = -1 \quad , \quad \tau(I)J = 0.$$

(ii) The trajectory associated with μ is an affine line parallel to I given by:

$$\mathfrak{T}(\mu) = \frac{MP}{C_1} + \mathrm{Vect}_{\mathbb{R}}(I) = \left\{\frac{MP}{C_1} + \mathfrak{s}I,\ \mathfrak{s}\in\mathbb{R}\right\}.$$

(iii) For all $X, X' \in \mathfrak{T}(\mu)$, we have $M_X = M_{X'}$.

(a) Suppose that $\mu \in \mathscr{M}at(\mathbf{I})$. For all $X \in \mathfrak{T}(\mu)$, we have:

$$M = sj_4(I,J) + m(X\tau(I) - I\tau(X)).$$

(b) Suppose that $\mu \in \mathscr{M}at(\mathbf{II})$. For all $X \in \mathfrak{T}(\mu)$, we have $M_X = 0$ and:

$$M = m(X\tau(I) - I\tau(X)).$$

Proof. (i) It suffices to address point (b). In this case, we have:

$$\tau(I)I = \frac{1}{m^2}\tau(P)P = \frac{C_2}{m^2} = 1$$

$$\tau(I)J = \frac{1}{sm^2}\tau(P)W = \frac{1}{sm^2}\begin{pmatrix}-E & p^T\end{pmatrix}\begin{pmatrix}l^Tp \\ p\wedge g + lE\end{pmatrix} = \frac{1}{sm^2}\left(-El^Tp + Ep^Tl\right) = 0$$

$$\tau(J)J = \frac{1}{s^2m^2}\tau(W)W = \frac{1}{-C_4}C_4 = -1$$

(ii) We have:

$$X \in \mathfrak{T}(\mu) \iff M_XP = 0$$

$$\begin{aligned}
&\iff (M + P\tau(X) - X\tau(P))P = 0\\
&\iff MP + P\tau(X)P - C_1X = 0\\
&\iff MP + P\tau(P)X - C_1X = 0\\
&\iff (\mathscr{E}) \ : \ \left(I_4 - \frac{P\tau(P)}{C_1}\right)X = \frac{MP}{C_1}.
\end{aligned}$$

Let $N := I_4 - P\tau(P)/C_1$. We have:

$$N^2 = \left(I_4 - \frac{P\tau(P)}{C_1}\right)^2 = I_4 - 2\frac{P\tau(P)}{C_1} + \frac{P\tau(P)P\tau(P)}{C_1^2} = N.$$

Thus, N is a vector projector and its image is:

$$\mathrm{Im}(N) = \mathrm{Ker}(N - I_4) = \mathrm{Ker}(P\tau(P)).$$

Therefore, $\mathrm{Im}(N)$ has dimension 3 (since $P\tau(P)$ is of rank 1). We have thus shown that $\mathfrak{T}(\mu)$ is an affine line.

It is easy to see that P is a solution of the homogeneous equation $(\mathscr{E}_0)$ associated with $(\mathscr{E})$ because:

$$\left(I_4 - \frac{P\tau(P)}{C_1}\right)P = P - \frac{PC_1}{C_1} = 0.$$

Since:

$$\tau(P)MP = \tau(\tau(P)MP) = -\tau(P)MP$$

i.e. $\tau(P)MP = 0$, we have:

$$\left(I_4 - \frac{P\tau(P)}{C_1}\right)\frac{MP}{C_1} = \frac{MP}{C_1} - \frac{P\tau(P)MP}{C_1^2} = \frac{MP}{C_1}.$$

Thus the result follows.

(iii) Let $X, X' \in \mathfrak{T}(\mu)$. There exist $u, u' \in \mathbb{R}$ such that:

$$X = \frac{MP}{C_1} + uI \quad , \quad X' = \frac{MP}{C_1} + u'I.$$

Thus, we have:

$$M_X - M_{X'} = (u - u')P\tau(P) - (u - u')P\tau(P) = 0.$$

(iv) (a) Since M_X is independent of $X \in \mathfrak{T}(\mu)$. Let us denote for $X_0 := MP/C_1 \in T$:

$$\Omega := \frac{1}{s}M_{X_0}.$$

We have $\Omega \in \mathscr{A}(1,3,\mathbb{R})$ and:

$$\begin{aligned}
\Omega I &= \frac{1}{sm}M_{X_0}P = 0\\
*(\Omega)I &= \frac{1}{sm} * (M_{X_0})P = \frac{1}{sm}W = J
\end{aligned}$$

Thus, by point (iii) of proposition C.6:

$$\Omega = j_4(I, J)$$

Therefore, we have:

$$M = M_{X_0} + \tau(P) - P\tau(X_0) = sj_4(I, J) + m(X_0\tau(I) - I\tau(X_0)).$$

(b) Let us denote for $X_0 := MP/C_1 \in T$:

$$\Omega := M_{X_0}.$$

We have $\Omega \in \mathscr{A}(1,3,\mathbb{R})$ and:

$$\Omega I = \frac{1}{m} M_{X_0} P = 0$$
$$*(\Omega) I = \frac{1}{m} * (M_{X_0}) P = \frac{1}{m} W = 0$$

Thus, by point (iii) of proposition C.6:

$$M_{X_0} = \Omega = j_4(I, 0) = 0.$$

By point (iii), M_X is independent of $X \in \mathfrak{T}(\mu)$ *i.e.* for all $X \in \mathfrak{T}(\mu)$, we have $M_X = M_{X_0} = 0$. Thus we have:

$$M = M_{X_0} + X_0 \tau(P) - P\tau(X_0) = m(X_0 \tau(I) - I\tau(X_0)).$$

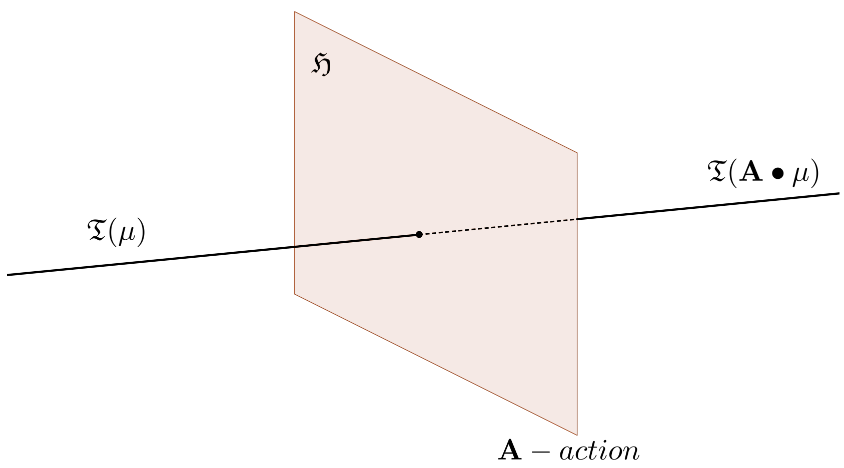
□

2.5 Interfaces with Action of the Poincaré Group

We conclude this text with an application within the framework of an interface in $\mathbb{R}^{1,3}$ where an action of the Poincaré group occurs.

We have shown in proposition 2.7 that for $\mathbf{A} \in \mathscr{P}oin$ and $\mu \in \mathscr{M}at(\mathbf{I} \cup \mathbf{II})$, the trajectory transforms by the simple formula:

$$\mathfrak{T}(\mathbf{A} \bullet \mu) = \mathbf{A} \bullet \mathfrak{T}(\mu).$$



Let us begin with the definition of an interface.

Definition 2.14

An **interface in** $\mathbb{R}^{1,3}$ is an affine hyperplane $\mathfrak{H}$ in $\mathbb{R}^{1,3}$ equipped with:

(i) a normal vector $n(\mathfrak{H}) := n \in \mathbb{R}^{1,3}$ and a point $H := H(\mathfrak{H}) \in \mathfrak{H}$ such that $\tau(n)n = -1$ (i.e., n is of *spatial type*), and:

$$\mathfrak{H} := \left\{ X \in \mathbb{R}^{1,3},\ n^T(X - H) = 0 \right\};$$

(ii) a matrix $\mathbf{A}(\mathfrak{H}) := \mathbf{A} \in \mathscr{P}oin$;

(iii) an **observer** $\mathscr{O}(\mathfrak{H}) := \mathscr{O}$ whose world line $\mathscr{L}(\mathscr{O})$ never intersects $\mathfrak{H}$.

We will denote:

$$\mathfrak{H}(n, H, \mathbf{A}, \mathscr{O}) := \mathfrak{H}.$$

The hyperplane $\mathfrak{H}$ divides $\mathbb{R}^{1,3}$ into two half-spaces. The world line of the observer $\mathscr{O}(\mathfrak{H})$ is in one of the two half-hyper-spaces. We denote $\mathfrak{H}_+$ as the half-space where it is located and $\mathfrak{H}_-$ as the other. We can choose two directions for n, the normal vector to $\mathfrak{H}$, and we assume that $H + n \in \mathfrak{H}_+$.

Lemma 2.15

Let a trajectory $\mathfrak{T} := X_0 + \mathrm{Vect}(U)$ be such that $U + H \notin \mathfrak{H}$, i.e., $n^T U \neq 0$. Then we have:

$$\mathfrak{T} \cap \mathfrak{H} = \left\{ X_0 + \frac{n^T(X_0 - H)}{n^T U} U \right\}.$$

Proof. For all $u \in \mathbb{R}$, we have:

$$X_0 + uU \in \mathfrak{H} \iff 0 = n^T(X_0 + uU - H) = n^T(X_0 - H) + un^T U \iff u = \frac{n^T(X_0 - H)}{n^T U}.$$

□

In particular, by theorem 2.13, for $\mu \in \mathscr{M}at(\mathbf{I} \cup \mathbf{II})$, we have:

$$\mathfrak{T}(\mu) = \frac{MP}{C_1} + \mathrm{Vect}_{\mathbb{R}}(I)$$

and thus:

$$\mathfrak{T}(\mu) \cap \mathfrak{H} = \left\{ X(\mathfrak{H}, \mu) := \frac{MP}{C_1} + \frac{n^T(\frac{MP}{C_1} - H)}{n^T I} I \right\} \tag{75}$$

We can make the following remarks and simplifications:

- Without loss of generality, by changing the observer whose world line is in $\mathfrak{H}_+$, we can assume that $\mathfrak{H}$ is stationary over time relative to the observer $\mathscr{O}$;
- By applying a Poincaré transformation to the reference frame, we can always reduce to the case where the hypersurface is such that $n := e_1^{(4)}$ and $H := 0$. In this frame, we thus have:

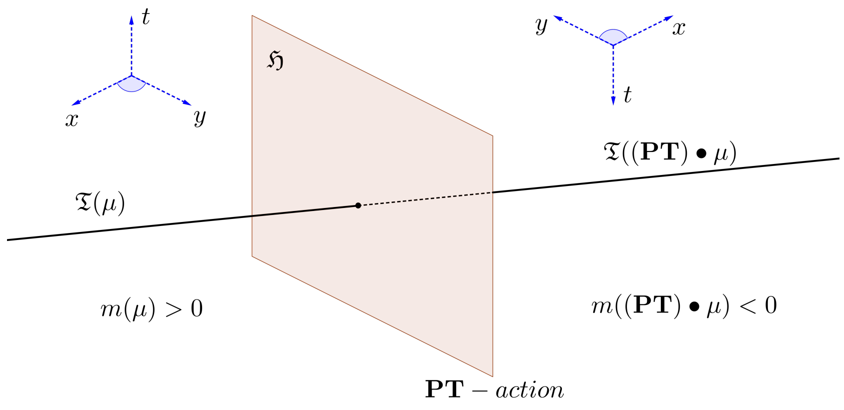
$$\mathfrak{H} = \left\{ X \in \mathbb{R}^{1,3},\ [X]_1 = 0 \right\}. \tag{76}$$

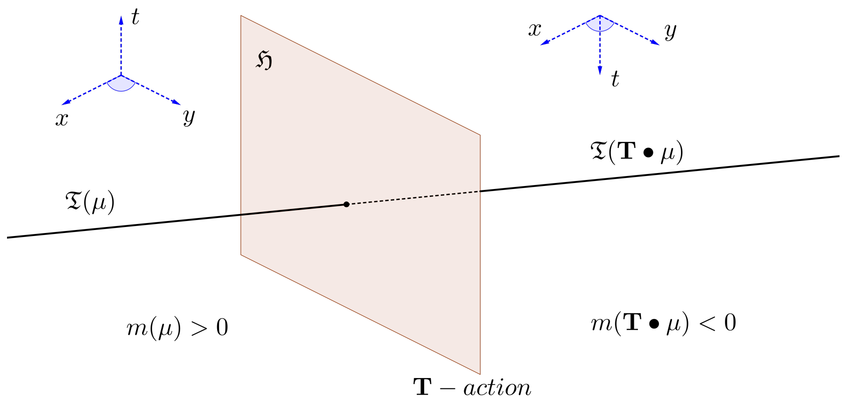
What happens when a particle coming from the $\mathfrak{H}_+$ side passes through the interface? We can consider two cases:

(A) The reference frame is the same on both sides. In this case, upon passing through the interface at the point $X(\mathfrak{H}, \mu)$, the particle finds itself at the point $\mathbf{A} \bullet X(\mathfrak{H}, \mu)$ with a direction $I(\mathbf{A} \bullet \mu)$. Thus, the particle is teleported.

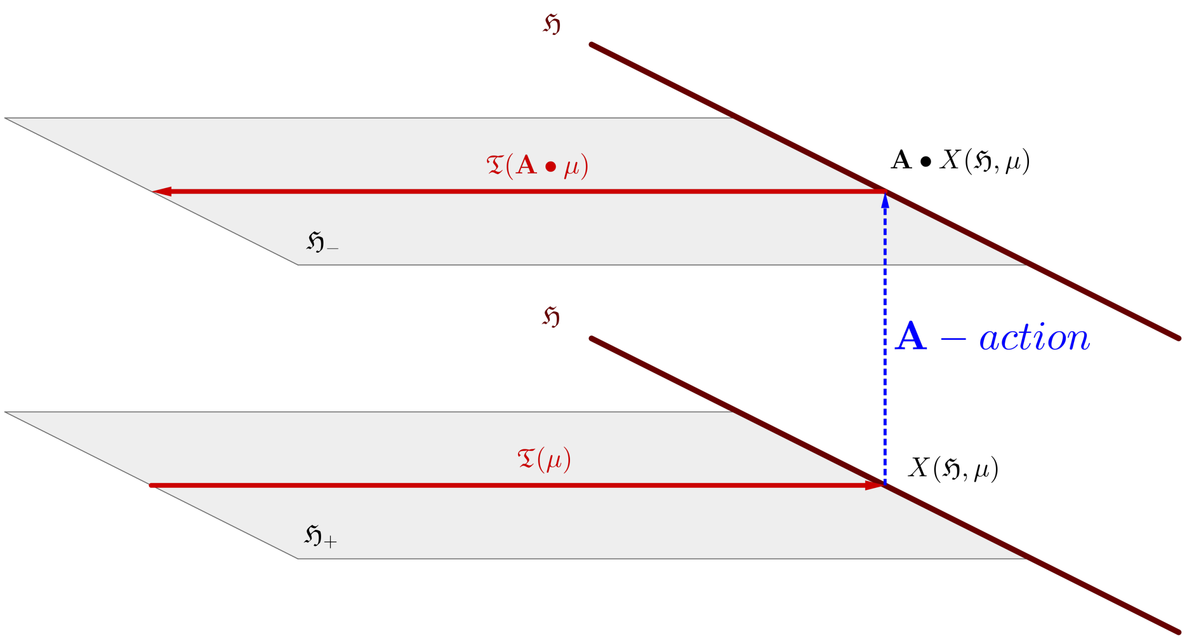
(B) The reference frame changes so that the trajectory of the matter remains continuous. In this case, the reference frame changes on the other side of the interface so that the particle does not change trajectory. The point $X(\mathfrak{H}, \mu)$ is identified by $\mathbf{A} \bullet X(\mathfrak{H}, \mu)$ using the reference frame on the $\mathfrak{H}_-$ side, and all points X on the $\mathfrak{H}_-$ side are identified for the observer $\mathscr{O}$ by $\mathbf{A} \bullet X$.

Here are two examples in case (B) of an interface where $\mathbf{A} := \mathbf{T}$ or $\mathbf{A} := \mathbf{PT}$:





We can then see that in both cases, the mass is reversed. To take up Jean-Pierre Petit's idea of spacetime covering, we will place ourselves in the case of a type (B) interface. The parts $\mathfrak{H}_+$ and $\mathfrak{H}_-$ are then seen as two half-spaces, one above the other:



A Skew-symmetric matrix

A.1 The space $\mathbb{R}^3$

Let us denote the canonical basis of $\mathbb{R}^3$:

$$\mathscr{B}ase\left(\mathbb{R}^3\right) := \left\{e_1 := \begin{pmatrix}1\\0\\0\end{pmatrix},\ e_2 := \begin{pmatrix}0\\1\\0\end{pmatrix},\ e_3 := \begin{pmatrix}0\\0\\1\end{pmatrix}\right\}. \tag{77}$$

Definition A.1

We equip $\mathbb{R}^3$ with the **scalar product** defined for all $u, v \in \mathbb{R}^3$ by $u^T v$ where $\bullet^T$ is the transpose of the matrices.
Let $u, v \in \mathbb{R}^3$. We say that u and v are **orthogonal** if $u^T v = 0$. We then write $u \perp v$.

For all $u := \sum_{i=1}^3 u^i e_i, v := \sum_{i=1}^3 v^i e_i \in \mathbb{R}^3$, we have:

$$u^T v = u^1 v^1 + u^2 v^2 + u^3 v^3 \tag{78}$$

$$u^T u = \left(u^1\right)^2 + \left(u^2\right)^2 + \left(u^3\right)^2 \geq 0 \tag{79}$$

We define the **norm of** u by:

$$\|u\| = \sqrt{u^T u}. \tag{80}$$

From this, we deduce the characterization of the dual of $\mathbb{R}^3$.

Proposition A.2

We have:

$$\left(\mathbb{R}^3\right)^* = \left\{v \longmapsto u^T v,\ u \in \mathbb{R}^3\right\}.$$

Proof. Let us prove the result by double inclusion. Define $\phi_u : v \longmapsto u^T v$ for all $u \in \mathbb{R}^3$. Clearly, $\phi_u \in \left(\mathbb{R}^3\right)^*$ for all $u \in \mathbb{R}^3$.

Let $\phi \in \left(\mathbb{R}^3\right)^*$. Define $u := \sum_{i=1}^3 \phi(e_i) e_i$. For all $j \in \{1, 2, 3\}$, we have:

$$\phi_u(e_j) = u^T e_j = \left(\sum_{i=1}^3 \phi(e_i) e_i\right)^T e_j = \sum_{i=1}^3 \phi(e_i) e_i^T e_j = \sum_{i=1}^3 \phi(e_i) \delta_{i,j} = \phi(e_j)$$

i.e. $\phi = \phi_u$ with $\delta_{i,j}$ being the **Kronecker symbol** equal to 1 if $i = j$ and 0 otherwise. Hence the result. □

A.2 Volume form and cross product in $\mathbb{R}^3$

Definition A.3

The **volume form of** $\mathbb{R}^3$ is defined as the alternating trilinear map:

$$\begin{array}{rccc} \mathrm{Vol}: & \left(\mathbb{R}^3\right)^3 & \longrightarrow & \mathbb{R} \\ & (u, v, w) & \longmapsto & \mathrm{Vol}(u, v, w) := \det(u, v, w) \end{array} \tag{81}$$

We have by expanding the determinant along the third column:

$$\begin{aligned} \mathrm{Vol}(u, v, w) &= \det(u, v, w) \\ &= w^1 \begin{vmatrix} u^2 & v^2 \\ u^3 & v^3 \end{vmatrix} - w^2 \begin{vmatrix} u^1 & v^1 \\ u^3 & v^3 \end{vmatrix} + w^3 \begin{vmatrix} u^1 & v^1 \\ u^2 & v^2 \end{vmatrix} \end{aligned}$$

$$= w^1\left(u^2v^3 - u^3v^2\right) + w^2\left(u^3v^1 - u^1v^3\right) + w^3\left(u^1v^2 - u^2v^1\right)$$
$$= \begin{pmatrix} w^1 & w^2 & w^3 \end{pmatrix} \begin{pmatrix} u^2v^3 - u^3v^2 \\ u^3v^1 - u^1v^3 \\ u^1v^2 - u^2v^1 \end{pmatrix}$$

Hence the following definition.

Definition A.4

The **cross product** of two vectors $u := \sum_{i=1}^3 u^i e_i, v := \sum_{i=1}^3 v^i e_i \in \mathbb{R}^3$ is defined by:

$$u \wedge v := \begin{pmatrix} u^2v^3 - u^3v^2 \\ u^3v^1 - u^1v^3 \\ u^1v^2 - u^2v^1 \end{pmatrix}.$$

Thus for all $u, v, w \in \mathbb{R}^3$:

$$(u \wedge v)^T w = w^T (u \wedge v) = \det(u, v, w) = \mathrm{Vol}(u, v, w). \tag{82}$$

For example, we have:

$$e_1 \wedge e_2 = e_3 \quad , \quad e_1 \wedge e_3 = -e_2 \quad , \tag{83}$$

We also denote by $\wedge$ the bilinear map:

$$\wedge : \begin{array}{ccc} \left(\mathbb{R}^3\right)^2 & \longrightarrow & \mathbb{R}^3 \\ (u, v) & \longmapsto & u \wedge v \end{array} \tag{84}$$

We have the following usual properties.

Proposition A.5

Let $u, v, w \in \mathbb{R}^3$.

(i) We have $u \wedge u = 0$.

(ii) (Antisymmetry) We have $u \wedge v = -v \wedge u$.

(iii) (Double cross product) We have:

$$u \wedge (v \wedge w) = \left(u^T w\right).v - \left(u^T v\right).w \quad , \quad (u \wedge v) \wedge w = \left(u^T w\right).v - \left(v^T w\right).u.$$

(iv) (Jacobi identity) We have:

$$u \wedge (v \wedge w) + w \wedge (u \wedge v) + v \wedge (w \wedge u) = 0.$$

Proof. (i) We have:

$$u \wedge u = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \wedge \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} = \begin{pmatrix} u^2u^3 - u^3u^2 \\ u^3u^1 - u^1u^3 \\ u^1u^2 - u^2u^1 \end{pmatrix} = 0.$$

(ii) We have:

$$u \wedge v = \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix} \wedge \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = \begin{pmatrix} u^2v^3 - u^3v^2 \\ u^3v^1 - u^1v^3 \\ u^1v^2 - u^2v^1 \end{pmatrix} = \begin{pmatrix} v^2u^3 - v^3u^2 \\ v^3u^1 - v^1u^3 \\ v^1u^2 - v^2u^1 \end{pmatrix} = -v \wedge u.$$

(iii) We have:

$$(u \wedge v) \wedge w = \begin{pmatrix} u^2v^3 - u^3v^2 \\ u^3v^1 - u^1v^3 \\ u^1v^2 - u^2v^1 \end{pmatrix} \wedge \begin{pmatrix} w^1 \\ w^2 \\ w^3 \end{pmatrix} = \begin{pmatrix} \left(u^3v^1 - u^1v^3\right) w^3 - \left(u^1v^2 - u^2v^1\right) w^2 \\ \left(u^1v^2 - u^2v^1\right) w^1 - \left(u^2v^3 - u^3v^2\right) w^3 \\ \left(u^2v^3 - u^3v^2\right) w^2 - \left(u^3v^1 - u^1v^3\right) w^1 \end{pmatrix}$$
$$= \begin{pmatrix} \left(u^1w^1 + u^2w^2 + u^3w^3\right) v^1 - \left(v^1w^1 + v^2w^2 + v^3w^3\right) u^1 \\ \left(u^1w^1 + u^2w^2 + u^3w^3\right) v^2 - \left(v^1w^1 + v^2w^2 + v^3w^3\right) u^2 \\ \left(u^1w^1 + u^2w^2 + u^3w^3\right) v^3 - \left(v^1w^1 + v^2w^2 + v^3w^3\right) u^3 \end{pmatrix}$$
$$= \left(u^1w^1 + u^2w^2 + u^3w^3\right) \begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} - \left(v^1w^1 + v^2w^2 + v^3w^3\right) \begin{pmatrix} u^1 \\ u^2 \\ u^3 \end{pmatrix}$$
$$= \left(u^T w\right).v - \left(v^T w\right).u$$

and we have:

$$u \wedge (v \wedge w) = -(v \wedge w) \wedge u = -\left(\left(u^T v\right).w - \left(u^T w\right).v\right) = \left(u^T w\right).v - \left(u^T v\right).w.$$

(iv) From point (iii) we have:

$$\begin{aligned} & u \wedge (v \wedge w) + w \wedge (u \wedge v) + v \wedge (w \wedge u) \\ = & \left(\left(u^T w\right).v - \left(u^T v\right).w\right) + \left(\left(w^T v\right).u - \left(w^T u\right).v\right) + \left(\left(v^T u\right).w - \left(v^T w\right).u\right) \\ = & \ 0 \end{aligned}$$

□

It forms a Lie bracket on $\mathbb{R}^3$.

Corollary A.6

The pair $\left(\mathbb{R}^3, \wedge\right)$ is a 3-dimensional Lie algebra.

A.3 The Space $\mathscr{A}(3, \mathbb{R})$

Definition A.7

We say that a matrix $A \in \mathscr{M}(3, \mathbb{R})$ is **antisymmetric** if:

$$A^T = -A$$

We denote $\mathscr{A}(3, \mathbb{R})$ as the vector space of antisymmetric matrices of size 3:

$$\mathscr{A}(3, \mathbb{R}) := \left\{A \in \mathscr{M}(3, \mathbb{R}),\ A^T = -A\right\}.$$

Let us define:

$$\forall A, B \in \mathscr{A}(3, \mathbb{R}),\ [A, B] := AB - BA. \tag{85}$$

We then have the following property.

Proposition A.8

The pair $(\mathscr{A}(3,\mathbb{R}),[])$ is a 3-dimensional Lie algebra with the canonical basis:

$$\mathscr{B}ase(\mathscr{A}(3,\mathbb{R})) := \left\{\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}, \begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}\right\}. \quad (86)$$

Proof. The map $(A,B)\in\mathscr{A}(3,\mathbb{R})^2 \longmapsto [A,B]$ is clearly bilinear, and $[A,A]=0$ for all $A\in\mathscr{A}(3,\mathbb{R})$. For all $A,B\in\mathscr{A}(3,\mathbb{R})$, we have:

$$[A,B]^T = (AB-BA)^T = B^TA^T - A^TB^T = -BA+AB = -[A,B]$$

i.e. $[A,B]\in\mathscr{A}(3,\mathbb{R})$. Furthermore, we have the Jacobi identity for all $A,B,C\in\mathscr{A}(3,\mathbb{R})$:

$$\begin{aligned}&[A,[B,C]]+[B,[C,A]]+[C,[A,B]]\\ =\ &(A(BC-CB)-(BC-CB)A)+(B(CA-AC)-(CA-AC)B)+(C(AB-BA)-(AB-BA)C)\\ =\ &0\end{aligned}$$

Let:

$$A := \begin{pmatrix}a&b&c\\d&e&f\\g&h&i\end{pmatrix}\in\mathscr{M}(3,\mathbb{R}).$$

We have:

$$\begin{aligned}A^T=-A &\iff \begin{pmatrix}a&d&g\\b&e&h\\c&f&i\end{pmatrix} = \begin{pmatrix}-a&-b&-c\\-d&-e&-f\\-g&-h&-i\end{pmatrix}\\ &\iff a=e=i=0 \ \wedge\ b=-d\ \wedge\ c=-g\ \wedge\ f=-h\\ &\iff A=\begin{pmatrix}0&-d&c\\d&0&-h\\-c&h&0\end{pmatrix} = h\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}+c\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}+d\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}.\end{aligned}$$

Hence the result. □

A.4 The Application j

Let the natural application be defined as:

$$\begin{array}{rccc} j: & \mathbb{R}^3 & \longrightarrow & \mathscr{A}(3,\mathbb{R}) \\ & \begin{pmatrix}x\\y\\z\end{pmatrix} & \longmapsto & \begin{pmatrix}0&-z&y\\z&0&-x\\-y&x&0\end{pmatrix}\end{array} \quad (87)$$

Proposition A.9

The application j induces a Lie algebra isomorphism:

$$j:(\mathbb{R}^3,\wedge)\longrightarrow(\mathscr{A}(3,\mathbb{R}),[]).$$

Proof. This application is clearly linear and injective. And since $\dim\mathscr{A}(3,\mathbb{R}) = \dim\mathbb{R}^3 = 3$, it is an isomorphism of vector spaces.

For all $u := \sum_{i=1}^3 u^ie_i, v := \sum_{i=1}^3 v^ie_i\in\mathbb{R}^3$, we have:

$$[j(u),j(v)] = j(u)j(v) - j(v)j(u)$$

$$= \begin{pmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & v^1u^2 - u^1v^2 & v^1u^3 - v^3u^1 \\ u^1v^2 - u^2v^1 & 0 & v^2u^3 - v^3u^2 \\ u^1v^3 - u^3v^1 & u^2v^3 - u^3v^2 & 0 \end{pmatrix}$$
$$= j\left(u \wedge v\right)$$

□

We recover the basis of proposition A.8:

$$\mathscr{B}ase\left(\mathscr{A}\left(3, \mathbb{R}\right)\right) = j\left(\mathscr{B}ase\left(\mathbb{R}^3\right)\right) = \left\{j\left(e_i\right),\ i \in \{1,2,3\}\right\}. \tag{88}$$

We then have the following properties.

Lemma A.10

Let $u, v \in \mathbb{R}^3$.

(i) We have $u \wedge v = j(u)\, v$

(ii) We have $j(u)^T = j(-u) = -j(u)$

(iii) We have $vu^T - v^T u I_3 = j(u)\, j(v)$

(iv) We have $j(u \wedge v) = [j(u), j(v)] = j(u)\, j(v) - j(v)\, j(u) = vu^T - uv^T$.

(v) Let $d \in \mathscr{M}(3, \mathbb{R})$. We have:
$$dj(u)\, d^T = j\left(\mathrm{Cof}(d)\, u\right)$$
with $\mathrm{Cof}(d)$ being the cofactor matrix of d.

Proof. (i) We have:

$$j(u)\, v = \begin{pmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} = u \wedge v.$$

(ii) We have:

$$j(u)^T = \begin{pmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{pmatrix}^T = \begin{pmatrix} 0 & u^3 & -u^2 \\ -u^3 & 0 & u^1 \\ u^2 & -u^1 & 0 \end{pmatrix} = j(-u) = -j(u).$$

(iii) We have:

$$vu^T - v^T u I_3 = \begin{pmatrix} v^1u^1 & v^1u^2 & v^1u^3 \\ v^2u^1 & v^2u^2 & v^2u^3 \\ v^3u^1 & v^3u^2 & v^3u^3 \end{pmatrix} - \left(v^1u^1 + v^2u^2 + v^2u^2\right) I_3$$
$$= \begin{pmatrix} -v^2u^2 - v^3u^3 & v^1u^2 & v^1u^3 \\ u^1v^2 & -v^1u^1 - v^3u^3 & v^2u^3 \\ u^1v^3 & u^2v^3 & -v^1u^1 - v^2u^2 \end{pmatrix}$$

$$j(u)\, j(v) = \begin{pmatrix} 0 & -u^3 & u^2 \\ u^3 & 0 & -u^1 \\ -u^2 & u^1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -v^3 & v^2 \\ v^3 & 0 & -v^1 \\ -v^2 & v^1 & 0 \end{pmatrix} = \begin{pmatrix} -v^2u^2 - v^3u^3 & v^1u^2 & v^1u^3 \\ u^1v^2 & -v^1u^1 - v^3u^3 & v^2u^3 \\ u^1v^3 & u^2v^3 & -v^1u^1 - v^2u^2 \end{pmatrix}$$

(iv) This follows from the fact that j is a Lie algebra morphism and from point (iii).

(v) Let $d \in \mathscr{M}(3,\mathbb{R})$ written in rows:

$$d := \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} := \begin{pmatrix} L_1^T \\ L_2^T \\ L_3^T \end{pmatrix}.$$

Since the cofactor matrix of d is given by:

$$\operatorname{Cof}(d) = \begin{pmatrix} \begin{vmatrix} d_{22} & d_{23} \\ d_{32} & d_{33} \end{vmatrix} & -\begin{vmatrix} d_{21} & d_{23} \\ d_{31} & d_{33} \end{vmatrix} & \begin{vmatrix} d_{21} & d_{22} \\ d_{31} & d_{32} \end{vmatrix} \\ -\begin{vmatrix} d_{12} & d_{13} \\ d_{32} & d_{33} \end{vmatrix} & \begin{vmatrix} d_{11} & d_{13} \\ d_{31} & d_{33} \end{vmatrix} & -\begin{vmatrix} d_{11} & d_{12} \\ d_{31} & d_{32} \end{vmatrix} \\ \begin{vmatrix} d_{12} & d_{13} \\ d_{22} & d_{23} \end{vmatrix} & -\begin{vmatrix} d_{11} & d_{13} \\ d_{21} & d_{23} \end{vmatrix} & \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} \end{pmatrix} = \begin{pmatrix} (L_2 \wedge L_3)^T \\ (L_3 \wedge L_1)^T \\ (L_1 \wedge L_2)^T \end{pmatrix}.$$

For all $i \in \{1,2,3\}$, we have:

$$\begin{aligned} dj(e_i)\,d^T &= \begin{pmatrix} L_1^T \\ L_2^T \\ L_3^T \end{pmatrix} j(e_i) \begin{pmatrix} L_1 & L_2 & L_3 \end{pmatrix} = \begin{pmatrix} L_1^T \\ L_2^T \\ L_3^T \end{pmatrix} \begin{pmatrix} e_i \wedge L_1 & e_i \wedge L_2 & e_i \wedge L_3 \end{pmatrix} \\ &= \begin{pmatrix} [L_1, e_i, L_1] & [L_1, e_i, L_2] & [L_1, e_i, L_3] \\ [L_2, e_i, L_1] & [L_2, e_i, L_2] & [L_2, e_i, L_3] \\ [L_3, e_i, L_1] & [L_3, e_i, L_2] & [L_3, e_i, L_3] \end{pmatrix} = \begin{pmatrix} 0 & -[L_2, e_i, L_1] & [L_1, e_i, L_3] \\ [L_2, e_i, L_1] & 0 & -[L_3, e_i, L_2] \\ -[L_3, e_i, L_1] & [L_3, e_i, L_2] & 0 \end{pmatrix} \\ &= j\begin{pmatrix} [L_3, e_i, L_2] \\ [L_1, e_i, L_3] \\ [L_2, e_i, L_1] \end{pmatrix} = j\begin{pmatrix} (L_2 \wedge L_3)^T e_i \\ (L_3 \wedge L_1)^T e_i \\ (L_1 \wedge L_2)^T e_i \end{pmatrix} = j(\operatorname{Cof}(d)\,e_i) \end{aligned}$$

Thus, by the linearity of j:

$$dj(u)\,d^T = \sum_{i=1}^{3} u^i dj(e_i)\,d^T = \sum_{i=1}^{3} u^i j(\operatorname{Cof}(d)\,e_i) = j\left(\sum_{i=1}^{3} u^i \operatorname{Cof}(d)\,e_i\right) = j(\operatorname{Cof}(d)\,u).$$

□

We thus deduce a representation of the dual of $\mathscr{A}(3,\mathbb{R})$.

Corollary A.11

We have:

$$\mathscr{A}(3,\mathbb{R})^* = \left\{N \longmapsto -\frac{1}{2}\operatorname{Tr}(MN),\ M \in \mathscr{A}(3,\mathbb{R})\right\}.$$

Proof. By the representation (A.2), the isomorphism j^{-1} induces an isomorphism on the duals:

$$\begin{array}{cccc} \left(j^{-1}\right)^* : & \left(\mathbb{R}^3\right)^* & \longrightarrow & \mathscr{A}(3,\mathbb{R})^* \\ & \phi_u : v \longrightarrow u^T v & \longmapsto & \psi_{j(u)} : j(v) \longrightarrow \phi_u(v) = u^T v \end{array}.$$

By point (iii) of lemma A.10, for all $M := j(u), N := j(v) \in \mathscr{A}(3,\mathbb{R})$ (with $u, v \in \mathbb{R}^3$), we have:

$$\psi_M(N) = \psi_{j(u)}(j(v)) = u^T v = -\frac{1}{2}\operatorname{Tr}(j(u)\,j(v)) = -\frac{1}{2}\operatorname{Tr}(MN).$$

Hence the result. □

For all $i, j \in \{1,2,3\}$:

$$\psi_{j(e_i)}(j(e_j)) = \phi_{e_i}(e_j) = e_i^T e_j = \delta_{i,j}. \tag{89}$$

B Skew-symmetric matrix in Minkowski space

B.1 The Applications τ

We define:

$$I_{1,0} := 1 \quad , \quad I_{1,3} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{90}$$

Definition B.1

For any matrix $A \in \mathscr{M}(1+k, 1+l, \mathbb{R})$ with $k, l \in \{0, 3\}$, we denote:

$$\tau(A) := I_{1,l} A^T I_{1,k}.$$

For example, we have:

- for any real number $x \in \mathbb{R}$:

$$\tau(x) = I_{1,0} x^T I_{1,0} = x; \tag{91}$$

- for any matrix $M := \begin{pmatrix} a & b^T \\ c & d \end{pmatrix} \in \mathscr{M}(1+3, \mathbb{R})$:

$$\tau(M) = I_{1,3} M^T I_{1,3} = \begin{pmatrix} a & -c^T \\ -b & d^T \end{pmatrix}; \tag{92}$$

- for any column matrix $X := \begin{pmatrix} t \\ r \end{pmatrix} \in \mathscr{M}(1+3, 1, \mathbb{R})$:

$$\tau(X) = I_{1,0} X^T I_{1,3} = X^T I_{1,3} = \begin{pmatrix} t & -r^T \end{pmatrix}; \tag{93}$$

- for any row matrix $Q := \begin{pmatrix} t & r^T \end{pmatrix} \in \mathscr{M}(1, 1+3, \mathbb{R})$:

$$\tau(Q) = I_{1,3} Q^T I_{1,0} = I_{1,3} Q^T = \begin{pmatrix} t \\ -r \end{pmatrix}. \tag{94}$$

Lemma B.2

Let $A \in \mathscr{M}(1+k, 1+l, \mathbb{R})$ and $B \in \mathscr{M}(1+l, 1+j, \mathbb{R})$ with $j, k, l \in \{0, 3\}$.

(i) We have $\tau(\tau(A)) = A$.

(ii) We have $\tau(AB) = \tau(B)\tau(A)$.

Proof. (i) We have:

$$\tau(\tau(A)) = \tau\left(I_{1,l} A^T I_{1,k}\right) = I_{1,k}\left(I_{1,l} A^T I_{1,k}\right)^T I_{1,l} = I_{1,k}\left(I_{1,k} A I_{1,l}\right) I_{1,l} = A.$$

(ii) We have:

$$\tau(AB) = I_{1,j}(AB)^T I_{1,k} = \left(I_{1,j} B^T I_{1,l}\right)\left(I_{1,l} A^T I_{1,k}\right) = \tau(B)\tau(A).$$

□

B.2 The space $\mathscr{A}(1, 3, \mathbb{R})$

Definition B.3

An **antisymmetric matrix of Minkowski space** is a matrix $M \in \mathscr{M}(4,\mathbb{R})$ such that :

$$\tau(M) = -M.$$

We denote their set as:

$$\mathscr{A}(1,3,\mathbb{R}) := \{M \in \mathscr{M}(4,\mathbb{R}),\ \tau(M) = -M\}.$$

We have the following simple representation.

Proposition B.4

We have the representation:

$$\mathscr{A}(1,3,\mathbb{R}) = \left\{ \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix},\ a,b \in \mathbb{R}^3 \right\}$$

Thus, $\mathscr{A}(1,3,\mathbb{R})$ is a vector subspace of $\mathscr{M}(4,\mathbb{R})$ of dimension 6 with the basis:

$$\mathscr{B}ase(\mathscr{A}(1,3,\mathbb{R})) = \left\{ \begin{pmatrix} 0 & e_i^T \\ e_i & 0 \end{pmatrix},\ i \in \{1,2,3\} \right\} \bigsqcup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & j(e_i) \end{pmatrix},\ i \in \{1,2,3\} \right\}.$$

Proof. Let $M := \begin{pmatrix} e & c \\ a & d \end{pmatrix} \in \mathscr{M}(1+3,\mathbb{R})$. We have:

$$\begin{aligned} M \in \mathscr{A}(1,3,\mathbb{R}) &\iff \tau(M) = -M \iff \begin{pmatrix} e & -a^T \\ -c^T & d^T \end{pmatrix} = \begin{pmatrix} -e & -c \\ -a & -d \end{pmatrix} \\ &\iff e = 0 \ \wedge\ c = a^T \ \wedge\ d := j(b) \in \mathscr{A}(3,\mathbb{R}) \iff M = \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} \\ &\iff M = \sum_{i=1}^{3} a^i \begin{pmatrix} 0 & e_i^T \\ e_i & 0 \end{pmatrix} + \sum_{i=1}^{3} b^i \begin{pmatrix} 0 & 0 \\ 0 & j(e_i) \end{pmatrix} \end{aligned}$$

with $a := \sum_{i=1}^3 a^i e_i$ and $b := \sum_{i=1}^3 b^i e_i$. □

Let us define the linear application $\ominus$:

$$\begin{aligned} \ominus :\ (\mathbb{R}^3)^2 &\longrightarrow \mathscr{A}(1,3,\mathbb{R}) \\ (a,b) &\longmapsto a \ominus b := \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} \end{aligned} \tag{95}$$

This is an isomorphism because it is clearly injective. We thus have:

$$\mathscr{A}(1,3,\mathbb{R}) = \left\{ a \ominus b,\ a,b \in \mathbb{R}^3 \right\}. \tag{96}$$

and:

$$\mathscr{B}ase(\mathscr{A}(1,3,\mathbb{R})) = \{e_i \ominus 0,\ i \in \{1,2,3\}\} \bigsqcup \{0 \ominus e_i,\ i \in \{1,2,3\}\}. \tag{97}$$

We equip $\mathscr{A}(1,3,\mathbb{R})$ with the natural Lie bracket on matrices as for $\mathscr{A}(3,\mathbb{R})$ (see (85) and proposition A.8):

$$\forall A,B \in \mathscr{A}(1,3,\mathbb{R}),\ [A,B] := AB - BA. \tag{98}$$

Lemma B.5

The bracket makes the pair $(\mathscr{A}(1,3,\mathbb{R}), [\ ,\ ])$ a Lie algebra.

Proof. Let $A, B, C \in \mathscr{A}(1,3,\mathbb{R})$. We have $[A,A] = 0$ and:

$$\tau([A,B]) = \tau(AB) - \tau(BA) = \tau(B)\tau(A) - \tau(A)\tau(B) = BA - AB = -[A,B]$$

i.e., $[A,B] \in \mathscr{A}(1,3,\mathbb{R})$. Thus, the map $(A,B) \longmapsto [A,B]$ is a bilinear map from $\mathscr{A}(1,3,\mathbb{R})^2$ to $\mathscr{A}(1,3,\mathbb{R})$. Furthermore, it satisfies the Jacobi identity, as in $\mathscr{A}(3,\mathbb{R})$ (see the proof of proposition A.8). □

We have the characterization of the dual of $\mathscr{A}(1,3,\mathbb{R})$.

Lemma B.6

We have:

$$\mathscr{A}(1,3,\mathbb{R})^* = \left\{\Lambda \longmapsto -\frac{1}{2}\mathrm{Tr}(M\Lambda),\ M \in \mathscr{A}(1,3,\mathbb{R})\right\}.$$

Proof. We prove by double inclusion:

$$(*)\ :\ \left(\left(\mathbb{R}^3\right)^2\right)^* = \left\{\Phi_{(a,b)} : \left(a',b'\right) \longrightarrow -a^T a' + b^T b',\ a,b \in \mathbb{R}^3\right\}.$$

Clearly, $\Phi_{(a,b)} \in \left(\left(\mathbb{R}^3\right)^2\right)^*$ for all $a, b \in \mathbb{R}^3$.

Let $\Phi \in \left(\left(\mathbb{R}^3\right)^2\right)^*$. Define:

$$a := -\sum_{k=1}^{3} \Phi(e_k, 0)\, e_k \quad , \quad b := \sum_{l=1}^{3} \Phi(0, e_l)\, e_l.$$

For all $i, j \in \{1,2,3\}$:

$$\begin{aligned}\Phi_{(a,b)}(e_i, e_j) &= -a^T e_i + b^T e_j = -\left(-\sum_{k=1}^{3}\Phi(e_k,0)\,e_k\right)^T e_i + \left(\sum_{l=1}^{3}\Phi(0,e_l)\,e_l\right)^T e_j \\ &= \sum_{k=1}^{3}\Phi(e_k,0)\,\delta_{k,i} + \sum_{l=1}^{3}\Phi(0,e_l)\,\delta_{l,j} = \Phi(e_i,0) + \Phi(0,e_j) = \Phi(e_i,e_j)\end{aligned}$$

i.e. $\Phi = \Phi_{(a,b)}$ Thus, the equality $(*)$ holds. Therefore, the isomorphism $\ominus^{-1}$ induces an isomorphism on the duals:

$$\begin{array}{cccc}\left(\ominus^{-1}\right)^* : & \left(\left(\mathbb{R}^3\right)^2\right)^* & \longrightarrow & \mathscr{A}(1,3,\mathbb{R})^* \\ & \Phi_{(a,b)} : (a',b') \longrightarrow -a^T a' + b^T b' & \longmapsto & \Psi_{a\ominus b} : a' \ominus b' \longrightarrow \Phi_{(a,b)}(a',b') = -a^T a' + b^T b'\end{array}.$$

Let $M := a \ominus b, N := a' \ominus b' \in \mathscr{A}(1,3,\mathbb{R})$ (with $a, b, a', b' \in \mathbb{R}^3$). Since:

$$\begin{aligned}-\frac{1}{2}\mathrm{Tr}(MN) &= -\frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix}0 & a^T \\ a & j(b)\end{pmatrix}\begin{pmatrix}0 & (a')^T \\ a' & j(b')\end{pmatrix}\right) = -\frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix}a^T a' & a^T j(b') \\ j(b)\,a & a(a')^T + j(b)\,j(b')\end{pmatrix}\right) \\ &= -\frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix}a^T a' & a^T j(b') \\ j(b)\,a & a(a')^T + (b')^T b - b^T b' I_3\end{pmatrix}\right) \\ &= -\frac{1}{2}\left(2a^T a' - 2b^T b'\right) = -a^T a' + b^T b'\end{aligned}$$

we have:

$$\Psi_M(N) = \Psi_{a\ominus b}\left(a' \ominus b'\right) = -a^T a' + b^T b' = -\frac{1}{2}\mathrm{Tr}\left((a\ominus b)\left(a'\ominus b'\right)\right) = -\frac{1}{2}\mathrm{Tr}(MN).$$

Hence the result. □

We then have for all $i, j \in \{1,2,3\}$:

$$\Psi_{e_i\ominus 0}(e_j \ominus 0) = -\delta_{i,j} \tag{99}$$

$$\Psi_{0\ominus e_i}(0 \ominus e_j) = \delta_{i,j} \tag{100}$$

$$\Psi_{0\ominus e_i}(e_j \ominus 0) = \Psi_{e_i\ominus 0}(0 \ominus e_j) = 0 \tag{101}$$

B.3 Cross product in $\mathbb{R}^{1,3}$

Definition B.7

The **volume form of** $\mathbb{R}^{1,3}$ is defined as the quadrilinear map:

$$\begin{array}{cccc} \mathrm{Vol}_4 : & \left(\mathbb{R}^{1,3}\right)^3 & \longrightarrow & \mathbb{R} \\ & (X_1, X_2, X_3, X_4) & \longmapsto & \mathrm{Vol}_4(X_1, X_2, X_3, X_4) := \det{}_4(X_1, X_2, X_3, X_4) \end{array}$$

For all $X_i := \begin{pmatrix} t_1 \\ r_i \end{pmatrix} \in \mathbb{R}^{1,3}$ ($i \in \{1,2,3,4\}$), by expanding the determinant along the first row:

$$\begin{aligned} \mathrm{Vol}_4(X_1, X_2, X_3, X_4) = \det{}_4(X_1, X_2, X_3, X_4) &= \begin{vmatrix} t_1 & t_2 & t_3 & t_4 \\ r_1 & r_2 & r_3 & r_4 \end{vmatrix} \\ &= t_1 \det(r_2, r_3, r_4) - t_2 \det(r_1, r_3, r_4) + t_3 \det(r_1, r_2, r_4) - t_4 \det(r_1, r_2, r_3) \\ &= t_4 r_1^T j(r_2)^T r_3 - r_4^T \left(t_3 j(r_2) r_1 + t_2 j(r_1) r_3 - t_1 j(r_2) r_3\right) \\ &= \begin{pmatrix} t_4 & -r_4^T \end{pmatrix} \begin{pmatrix} r_1^T j(r_2)^T r_3 \\ t_3 j(r_2) r_1 + t_2 j(r_1) r_3 - t_1 j(r_2) r_3 \end{pmatrix} \\ &= \begin{pmatrix} t_4 & -r_4^T \end{pmatrix} \begin{pmatrix} 0 & r_1^T j(r_2)^T \\ j(r_2) r_1 & j(r_1 t_2 - r_2 t_1) \end{pmatrix} \begin{pmatrix} t_3 \\ r_3 \end{pmatrix} \\ &= \tau(X_4) \begin{pmatrix} 0 & r_1^T j(r_2)^T \\ j(r_2) r_1 & j(r_1 t_2 - r_2 t_1) \end{pmatrix} X_3 \end{aligned}$$

Hence the following definitions[4].

Definition B.8

(i) The **application** j_4 is defined as:

$$\begin{array}{cccc} j_4 : & \left(\mathbb{R}^{1,3}\right)^2 & \longrightarrow & \mathscr{A}(1,3,\mathbb{R}) \\ & \left(\begin{pmatrix} t_1 \\ r_1 \end{pmatrix}, \begin{pmatrix} t_2 \\ r_2 \end{pmatrix}\right) & \longmapsto & \begin{pmatrix} 0 & r_1^T j(r_2)^T \\ j(r_2) r_1 & j(r_1 t_2 - r_2 t_1) \end{pmatrix} \end{array}$$

(ii) Let $X_1, X_2, X_3 \in \mathbb{R}^{1,3}$. The **cross product of X_1, X_2, and X_3 in $\mathbb{R}^{1,3}$** is defined as:

$$X_1 \wedge X_2 \wedge X_3 := j_4(X_1, X_2) X_3.$$

We thus obtain a formula similar to (82) for $\mathbb{R}^3$. For all $X_1, X_2, X_3, X_4 \in \mathbb{R}^{1,3}$, we have:

$$\mathrm{Vol}_4(X_1, X_2, X_3, X_4) = \tau(X_4)(X_1 \wedge X_2 \wedge X_3) = \tau(X_1 \wedge X_2 \wedge X_3) X_4. \tag{102}$$

Thus, for all $\begin{pmatrix} t_1 \\ r_1 \end{pmatrix}, \begin{pmatrix} t_2 \\ r_2 \end{pmatrix} \in \mathbb{R}^{1,3}$:

$$j_4\left(\begin{pmatrix} t_1 \\ r_1 \end{pmatrix}, \begin{pmatrix} t_2 \\ r_2 \end{pmatrix}\right) = \begin{pmatrix} 0 & (r_2 \wedge r_1)^T \\ r_2 \wedge r_1 & j(r_1 t_2 - r_2 t_1) \end{pmatrix} \tag{103}$$

Since:

$$j_4(X_1, X_2) = \begin{pmatrix} 0 & r_1^T j(r_2)^T \\ j(r_2) r_1 & j(r_1 t_2 - r_2 t_1) \end{pmatrix} = -\begin{pmatrix} 0 & r_2^T j(r_1)^T \\ j(r_1) r_2 & j(r_2 t_1 - r_1 t_2) \end{pmatrix} = -j_4(X_2, X_1)$$

[4] In Chapter 13 of [17], the volume is defined by $\mathrm{Vol}_4(X_1, X_2, X_3, X_4) = \begin{vmatrix} r_1 & r_2 & r_3 & r_4 \\ t_1 & t_2 & t_3 & t_4 \end{vmatrix}$ which explains the "−" sign between the two definitions of j_4.

we have:

$$X_1 \wedge X_2 \wedge X_3 = j_4\left(X_1, X_2\right) X_3 = -j_4\left(X_2, X_1\right) X_3 = -X_2 \wedge X_1 \wedge X_3 \tag{104}$$

The application j_4 is a bilinear antisymmetric map and is non-injective because for all $X_1, X_2 \in \mathbb{R}^{1,3}$:

$$j_4\left(X_1, X_2\right) = 0 \iff r_2 \wedge r_1 = 0 \ \wedge \ r_1 t_2 - r_2 t_1 = 0 \iff \exists \alpha, \beta \in \mathbb{R},\ \alpha X_1 + \beta X_2 = 0. \tag{105}$$

Thus, as the determinant $\det_4$ is a 4-linear alternating form, we have, for example:

$$\begin{aligned}
\tau\left(X_1 \wedge X_2 \wedge X_3\right) X_1 = 0 \quad & i.e.\ \ X_1 \perp_4 \left(X_1 \wedge X_2 \wedge X_3\right) \\
\tau\left(X_1 \wedge X_2 \wedge X_3\right) X_2 = 0 \quad & i.e.\ \ X_2 \perp_4 \left(X_1 \wedge X_2 \wedge X_3\right) \\
\tau\left(X_1 \wedge X_2 \wedge X_3\right) X_3 = 0 \quad & i.e.\ \ X_3 \perp_4 \left(X_1 \wedge X_2 \wedge X_3\right)
\end{aligned}$$

C The dual operator

C.1 Generalities

Definition C.1

The **Hodge dual operator on** $\mathscr{A}(1,3,\mathbb{R})$ is the linear application defined by:

$$\begin{array}{rccc} *: & \mathscr{A}(1,3,\mathbb{R}) & \longrightarrow & \mathscr{A}(1,3,\mathbb{R}) \\ & \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} & \longmapsto & \begin{pmatrix} 0 & b^T \\ b & j(-a) \end{pmatrix} \end{array}$$

Thus, for all $a := \sum_{i=1}^3 a^i e_i, b := \sum_{i=1}^3 a^i e_i \in \mathbb{R}^3$:

$$*(a \ominus b) = * \begin{pmatrix} 0 & a^1 & a^2 & a^3 \\ a^1 & 0 & -b^3 & b^2 \\ a^2 & b^3 & 0 & -b^1 \\ a^3 & -b^2 & b^1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b^1 & b^2 & b^3 \\ b^1 & 0 & a^3 & -a^2 \\ b^2 & -a^3 & 0 & a^1 \\ b^3 & a^2 & -a^1 & 0 \end{pmatrix} = b \ominus (-a). \tag{106}$$

The application $*$ is an automorphism of $\mathscr{A}(1,3,\mathbb{R})$ because we have:

$$\forall M := \begin{pmatrix} 0 & a^T \\ a & j(-b) \end{pmatrix} \in \mathscr{A}(1,3,\mathbb{R}), \quad *(*(M)) = * \begin{pmatrix} 0 & a^T \\ a & j(-b) \end{pmatrix} = \begin{pmatrix} 0 & -b^T \\ -b & j(-a) \end{pmatrix} = -M. \tag{107}$$

We have the following properties.

Proposition C.2

(i) For all $M \in \mathscr{A}(1,3,\mathbb{R})$ and all $X \in \mathbb{R}^{1,3}$, the vector $*(M)X$ is orthogonal to X.

(ii) For all $M \in \mathscr{A}(1,3,\mathbb{R})$, we have:

$$(*(M))^2 = M^2 - \frac{1}{2}\operatorname{Tr}\left(M^2\right) I_4.$$

(iii) For all $M \in \mathscr{A}(1,3,\mathbb{R})$ and all $Y \in \mathbb{R}^{1,3}$, the mapping:

$$Y \in \mathbb{R}^{1,3} \longmapsto *(M + X\tau(Y) - Y\tau(X))X$$

is constant at $*(M)X$.

Proof. (i) Let us set $Y := *(M)X$. Since $*(M) \in \mathscr{A}(1,3,\mathbb{R})$ and $\tau(X)Y \in \mathbb{R}$, we have:

$$\tau(X)Y = \tau(\tau(X)Y) = \tau(X)\tau(*(M))X = \tau(X)(-*(M))X = -\tau(X)Y$$

i.e., $\tau(X)Y = 0$.

(ii) For any $M := \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} \in \mathscr{A}(1,3,\mathbb{R})$:

$$\begin{aligned} M^2 - \frac{1}{2}\operatorname{Tr}\left(M^2\right) I_4 &= \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix}\begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} - \frac{1}{2}\operatorname{Tr}\left(M^2\right) I_4 \\ &= \begin{pmatrix} a^T a & a^T j(b) \\ j(b)a & aa^T + j(b)^2 \end{pmatrix} - \frac{a^T a + \operatorname{Tr}\left(aa^T + j(b)^2\right)}{2} I_4 \\ &= \begin{pmatrix} a^T a & -b^T j(a) \\ -j(a)b & aa^T + j(b)^2 \end{pmatrix} - a^T a I_4 - \frac{\operatorname{Tr}\left(bb^T - b^T b I_3\right)}{2} I_4 \end{aligned}$$

$$= \begin{pmatrix} 0 & -b^T j(a) \\ -j(a) b & aa^T - a^T a I_3 + j(b)^2 \end{pmatrix} - b^T b I_4 = \begin{pmatrix} b^T b & -b^T j(a) \\ -j(a) b & bb^T + j(a)^2 \end{pmatrix}$$
$$= \begin{pmatrix} 0 & b^T \\ b & -j(a) \end{pmatrix} \begin{pmatrix} 0 & b^T \\ b & -j(a) \end{pmatrix} = (*(M))^2$$

(iii) Let us define:

$$X := \begin{pmatrix} t \\ r \end{pmatrix} \quad , \quad Y := \begin{pmatrix} t' \\ r' \end{pmatrix} \quad , \quad M := \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix}.$$

We have:

$$*(M) X = \begin{pmatrix} 0 & b^T \\ b & j(-a) \end{pmatrix} \begin{pmatrix} t \\ r \end{pmatrix} = \begin{pmatrix} b^T r \\ j(-a) r + tb \end{pmatrix} = \begin{pmatrix} b^T r \\ r \wedge a + tb \end{pmatrix}$$

and:

$$M + X\tau(Y) - Y\tau(X) = \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} + \begin{pmatrix} t \\ r \end{pmatrix} \begin{pmatrix} t' & -(r')^T \end{pmatrix} - \begin{pmatrix} t' \\ r' \end{pmatrix} \begin{pmatrix} t & -r^T \end{pmatrix}$$
$$= \begin{pmatrix} 0 & a^T + t' r^T - t(r')^T \\ a + t'r - tr' & j(b) + r' r^T - r(r')^T \end{pmatrix}$$
$$= \begin{pmatrix} 0 & a^T + t' r^T - t(r')^T \\ a + t'r - tr' & j(b + r \wedge r') \end{pmatrix}$$

Thus we have:

$$*(M + X\tau(Y) - Y\tau(X)) X = \begin{pmatrix} 0 & b^T + (r \wedge r')^T \\ b + r \wedge r' & j(a + t'r - tr') \end{pmatrix} \begin{pmatrix} t \\ r \end{pmatrix}$$
$$= \begin{pmatrix} b^T r \\ r \wedge a + tb \end{pmatrix} = *(M) X$$

is therefore independent of X.

□

Lemma C.3

Let $M \in \mathscr{A}(1,3,\mathbb{R})$ and $X \in \mathbb{R}^{1,3}$ non-zero. Define $Y := *(M) X$. Suppose that $\tau(X) X = \tau(Y) Y = 0$. Then $Y \in \text{Vect}_{\mathbb{R}}(X)$.

Proof. Let us decompose the vectors X and Y in $\mathbb{R}^{1,3}$:

$$X := \begin{pmatrix} t \\ r \end{pmatrix} \quad , \quad Y := \begin{pmatrix} t' \\ r' \end{pmatrix}.$$

From point (i) of proposition C.2, we have $\tau(X) Y = 0$, i.e.:

$$(i) \qquad 0 = \tau(X) X = t^2 - r^T r$$
$$(ii) \qquad 0 = \tau(Y) Y = (t')^2 - (r')^T r' = 0$$
$$(ii) \qquad 0 = \tau(X) Y = tt' - r^T r'$$

Thus, we have from points (i) and (ii):

$$0 = t(tt' - r^T r') = r^T r t' - t r^T r' = r^T (t'r - tr')$$
$$0 = t'(tt' - r^T r') = t(r')^T r' - t' r^T r' = -(r')^T (t'r - tr')$$

Hence, $r, r' \in \mathrm{Vect}\,(t'r - tr')^{\perp}$ (in $\mathbb{R}^3$ with the usual scalar product) and $t'r - tr' \in \mathrm{Vect}\,(r, r')$. Therefore, $t'r - tr' = 0$. From point (i), we have $t \neq 0$ (since otherwise $r = 0$ and thus $X = 0$, contradicting the assumption), and therefore:

$$r' = \frac{t'}{t} r,$$

and thus we obtain:

$$Y = \begin{pmatrix} t' \\ r' \end{pmatrix} = \frac{t'}{t} \begin{pmatrix} t \\ r \end{pmatrix} = \frac{t'}{t} X \in \mathrm{Vect}_{\mathbb{R}}\,(X)\,.$$

□

C.2 Hodge operator and Lorentz group

Let us begin with a lemma on the matrices of $\mathscr{L}or_n$.

Lemma C.4

Let $\mathbf{L}_n := \begin{pmatrix} a & b^T \\ c & d \end{pmatrix} \in \mathscr{L}or_n$.

(i) We have:

$$\det\left(ad - cb^T\right) = a^2.$$

(ii) We have:

$$1 = a^2 - c^T c \qquad ac = db$$
$$I_3 = dd^T - cc^T \qquad ab = d^T c$$

(iii) We have $\det\,(d) = a$ and:

$$\mathrm{Cof}\,(d) = ad - cb^T.$$

Proof. (i) Since $a \neq 0$, by Schur complement, we have:

$$\mathbf{L}_n = \begin{pmatrix} a & b^T \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c/a & I_3 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & d - \frac{1}{a} cb^T \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{a} b^T \\ 0 & I_3 \end{pmatrix}$$

Thus:

$$1 = \det\,(\mathbf{L}_n) = a \det\left(d - \frac{1}{a} cb^T\right) = a \det\left(\left(\frac{1}{a} I_3\right) \times \left(ad - cb^T\right)\right) = \frac{1}{a^2} \det\left(ad - cb^T\right).$$

(ii) Since $\mathbf{L}_n \in \mathscr{L}or_n$, we have:

$$\begin{pmatrix} 1 & 0 \\ 0 & I_3 \end{pmatrix} = \begin{pmatrix} a & b^T \\ c & d \end{pmatrix} \begin{pmatrix} a & -c^T \\ -b & d^T \end{pmatrix} = \begin{pmatrix} a^2 - b^T b & -ac^T + b^T d^T \\ ac - db & dd^T - cc^T \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ 0 & I_3 \end{pmatrix} = \begin{pmatrix} a & -c^T \\ -b & d^T \end{pmatrix} \begin{pmatrix} a & b^T \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 - c^T c & ab^T - c^T d \\ -ab + d^T c & d^T d - bb^T \end{pmatrix}$$

Thus, we have the relations:

$$1 = a^2 - c^T c = a^2 - b^T b \qquad ac = db$$
$$I_3 = dd^T - cc^T = d^T d - bb^T \qquad ab = d^T c$$

(iii) We have:

$$d\left(ad^T - bc^T\right) = add^T - dbc^T = a\left(I_3 + cc^T\right) - dbc^T = aI_3 + (ac - db)\,c^T = aI_3.$$

Thus, from (i), we have:

$$a^3 = \det(d)\det\left(ad^T - bc^T\right) = \det(d)\,a^2$$

i.e., we have $\det(d) = a$. Thus:

$$\mathrm{Cof}(d) = ad - cb^T.$$

Thus, we obtain the lemma. □

We deduce the following important proposition.

Proposition C.5

For all $M \in \mathscr{A}(1,3,\mathbb{R})$ and all $\mathbf{L} \in \mathscr{L}or$:

$$*\left(\mathbf{L}M\tau(\mathbf{L})\right) = \det(\mathbf{L})\,\mathbf{L} * (M)\,\tau(\mathbf{L}).$$

Proof. We treat two cases.

(1) Case $\mathbf{L} := \mathbf{P}^\nu\mathbf{T}^\lambda$. For all $M := \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} \in \mathscr{A}(1,3,\mathbb{R})$, we have:

$$\begin{aligned}
*\left(\mathbf{L}M\tau(\mathbf{L})\right) &= *\left(\mathbf{P}^\nu\mathbf{T}^\lambda \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} \mathbf{T}^\lambda\mathbf{P}^\nu\right) \\
&= *\begin{pmatrix} 0 & (-1)^{\lambda+\nu} a^T \\ (-1)^{\lambda+\nu} a & j(b) \end{pmatrix} = \begin{pmatrix} 0 & b^T \\ b & (-1)^{\lambda+\nu} j(-a) \end{pmatrix} \\
\det(\mathbf{L})\,\mathbf{L} * (M)\,\tau(\mathbf{L}) &= \det\left(\mathbf{P}^\nu\mathbf{T}^\lambda\right)\mathbf{P}^\nu\mathbf{T}^\lambda * \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} \tau\left(\mathbf{P}^\nu\mathbf{T}^\lambda\right) \\
&= (-1)^{\lambda+3\nu}\,\mathbf{P}^\nu\mathbf{T}^\lambda \begin{pmatrix} 0 & b^T \\ b & j(-a) \end{pmatrix} \mathbf{T}^\lambda\mathbf{P}^\nu \\
&= (-1)^{\lambda+\nu} \begin{pmatrix} 0 & (-1)^{\lambda+\nu} b^T \\ (-1)^{\lambda+\nu} b & j(-a) \end{pmatrix} = \begin{pmatrix} 0 & b^T \\ b & (-1)^{\lambda+\nu} j(-a) \end{pmatrix}
\end{aligned}$$

Thus, we obtain the result in this case.

(2) Case $\mathbf{L} := \mathbf{L}_n := \begin{pmatrix} a & b^T \\ c & d \end{pmatrix} \in \mathscr{L}or_n$. We apply the results of lemma C.4.

Since the result is linear in M, it suffices to show the result on the basis (97) of $\mathscr{A}(1,3,\mathbb{R})$. For all $i \in \{1,2,3\}$, using the lemma, we have:

$$\begin{aligned}
\mathbf{L}_n * (0 \ominus e_i)\,\tau(\mathbf{L}_n) &= \begin{pmatrix} a & b^T \\ c & d \end{pmatrix}\begin{pmatrix} 0 & e_i^T \\ e_i & 0 \end{pmatrix}\begin{pmatrix} a & -c^T \\ -b & d^T \end{pmatrix} = \begin{pmatrix} b^Te_i & ae_i^T \\ de_i & ce_i^T \end{pmatrix}\begin{pmatrix} a & -c^T \\ -b & d^T \end{pmatrix} \\
&= \begin{pmatrix} 0 & ae_i^Td^T - b^Te_ic^T \\ ade_i - ce_i^Tb & ce_i^Td^T - de_ic^T \end{pmatrix} = \begin{pmatrix} 0 & e_i^T\left(ad^T - bc^T\right) \\ \left(ad - cb^T\right)e_i & j\left((de_i)\wedge c\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & e_i^T\,\mathrm{Cof}(d)^T \\ \mathrm{Cof}(d)\,e_i & j\left(j(de_i)\,c\right) \end{pmatrix} = \begin{pmatrix} 0 & e_i^T\,\mathrm{Cof}(d)^T \\ \mathrm{Cof}(d)\,e_i & j\left(-j(c)\,de_i\right) \end{pmatrix} \\
*\left(\mathbf{L}_n(0 \ominus e_i)\,\tau(\mathbf{L}_n)\right) &= *\left(\begin{pmatrix} a & b^T \\ c & d \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & j(e_i) \end{pmatrix}\begin{pmatrix} a & -c^T \\ -b & d^T \end{pmatrix}\right) = *\left(\begin{pmatrix} 0 & b^Tj(e_i) \\ 0 & dj(e_i) \end{pmatrix}\begin{pmatrix} a & -c^T \\ -b & d^T \end{pmatrix}\right) \\
&= *\begin{pmatrix} -b^Tj(e_i)\,b & b^Tj(e_i)\,d^T \\ -dj(e_i)\,b & dj(e_i)\,d^T \end{pmatrix} = *\begin{pmatrix} 0 & \frac{1}{a}c^Tdj(e_i)\,d^T \\ -\frac{1}{a}dj(e_i)\,d^Tc & j\left(\mathrm{Cof}(d)\,e_i\right) \end{pmatrix} \\
&= \begin{pmatrix} 0 & e_i^T\,\mathrm{Cof}(d)^T \\ \mathrm{Cof}(d)\,e_i & j\left(\frac{1}{a}dj(e_i)\,d^Tc\right) \end{pmatrix} = \begin{pmatrix} 0 & e_i^T\,\mathrm{Cof}(d)^T \\ \mathrm{Cof}(d)\,e_i & j\left(\frac{1}{a}j\left(\mathrm{Cof}(d)\,e_i\right)c\right) \end{pmatrix}
\end{aligned}$$

$$= \begin{pmatrix} 0 & e_i^T \mathrm{Cof}(d)^T \\ \mathrm{Cof}(d) e_i & j\left(-j(c) \frac{1}{a} \mathrm{Cof}(d) e_i\right) \end{pmatrix} = \begin{pmatrix} 0 & e_i^T \mathrm{Cof}(d)^T \\ \mathrm{Cof}(d) e_i & j(-j(c) d e_i) \end{pmatrix}.$$

because:

$$j(c) d = \frac{1}{a} j(c) \left(\mathrm{Cof}(d) + cb^T\right) = \frac{1}{a} j(c) \mathrm{Cof}(d) + \frac{1}{a} j(c) cb^T = \frac{1}{a} j(c) \mathrm{Cof}(d).$$

Thus, applying the operator $*$ on these two relations, we have for all $i \in \{1,2,3\}$:

$$\begin{aligned} *(\mathbf{L}_n (e_i \ominus 0) \tau(\mathbf{L}_n)) &= *(\mathbf{L}_n * (0 \ominus e_i) \tau(\mathbf{L}_n)) \\ &= *(*(\mathbf{L}_n (0 \ominus e_i) \tau(\mathbf{L}_n))) \\ &= -\mathbf{L}_n (0 \ominus e_i) \tau(\mathbf{L}_n) \\ &= \mathbf{L}_n * (e_i \ominus 0) \tau(\mathbf{L}_n) \end{aligned}$$

Thus, we obtain the result in this case as well.

We will deduce the general case from cases (1) and (2). For any $\mathbf{L} := \mathbf{P}^\nu \mathbf{T}^\lambda \mathbf{L}_n$, we have with $M' := \mathbf{L}_n M \tau(\mathbf{L}_n)$:

$$\begin{aligned} *(\mathbf{L} M \tau(\mathbf{L})) &= *\left(\mathbf{P}^\nu \mathbf{T}^\lambda M' \mathbf{T}^\lambda \mathbf{P}^\nu\right) \\ &= \det\left(\mathbf{P}^\nu \mathbf{T}^\lambda\right) \mathbf{P}^\nu \mathbf{T}^\lambda * \left(M'\right) \mathbf{T}^\lambda \mathbf{P}^\nu \\ &= \det\left(\mathbf{P}^\nu \mathbf{T}^\lambda\right) \mathbf{P}^\nu \mathbf{T}^\lambda * (\mathbf{L}_n M \tau(\mathbf{L}_n)) \mathbf{T}^\lambda \mathbf{P}^\nu \\ &= \det\left(\mathbf{P}^\nu \mathbf{T}^\lambda\right) \mathbf{P}^\nu \mathbf{T}^\lambda \det(\mathbf{L}_n) \mathbf{L}_n * (M) \tau(\mathbf{L}_n) \mathbf{T}^\lambda \mathbf{P}^\nu \\ &= \det(\mathbf{L}) \mathbf{L} * (M) \tau(\mathbf{L}) \end{aligned}$$

Thus, the result follows. □

C.3 Links between the map j_4 and the dual operator

The following property states some useful links between the operator $*$ and the map j_4.

Proposition C.6

(i) For all $X_1, X_2 \in \mathbb{R}^{1,3}$, we have:

$$*(X_2 \tau(X_1) - X_1 \tau(X_2)) = j_4(X_1, X_2)$$

(ii) For all $X_1, X_2 \in \mathbb{R}^{1,3}$ and any $M \in \mathscr{A}(1,3,\mathbb{R})$, we have:

$$\tau(X_1) * (M) X_2 = \frac{1}{2} \mathrm{Tr}(j_4(X_1, X_2) M).$$

(iii) Let $M \in \mathscr{A}(1,3,\mathbb{R})$ and $X \in \mathbb{R}^{1,3}$ such that:

$$\tau(X) X = 1, \quad MX = 0.$$

Then we have:

$$M = j_4(*(M) X, X).$$

Proof. (i) For all $X_1 := \begin{pmatrix} t_1 \\ r_1 \end{pmatrix}, X_2 := \begin{pmatrix} t_2 \\ r_2 \end{pmatrix} \in \mathbb{R}^{1,3}$, we have:

$$*(X_2 \tau(X_1) - X_1 \tau(X_2)) = *\left(\begin{pmatrix} t_2 t_1 & -t_2 r_1^T \\ t_1 r_2 & -r_2 r_1^T \end{pmatrix} - \begin{pmatrix} t_1 t_2 & -t_1 r_2^T \\ t_2 r_1 & -r_1 r_2^T \end{pmatrix}\right) = *\begin{pmatrix} 0 & t_1 r_2^T - t_2 r_1^T \\ t_1 r_2 - t_2 r_1 & r_1 r_2^T - r_2 r_1^T \end{pmatrix}$$

$$= \begin{pmatrix} 0 & r_1^T j(r_2)^T \\ j(r_2) r_1 & j(t_2 r_1 - t_1 r_2) \end{pmatrix} = j_4(X_1, X_2)$$

(ii) We have for all $M := \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} \in \mathscr{A}(1,3,\mathbb{R})$:

$$\begin{aligned}
\frac{1}{2}\mathrm{Tr}\left(j_4(X_1,X_2)M\right) &= \frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix} 0 & r_1^T j(r_2)^T \\ j(r_2)r_1 & j(t_2r_1-t_1r_2)\end{pmatrix}\begin{pmatrix} 0 & a^T \\ a & j(b)\end{pmatrix}\right) \\
&= \frac{1}{2}\mathrm{Tr}\left(\begin{pmatrix} r_1^T j(r_2)^T a & r_1^T j(r_2)^T j(b) \\ j(t_2r_1-t_1r_2)a & j(r_2)r_1a^T + j(t_2r_1-t_1r_2)j(b)\end{pmatrix}\right) \\
&= t_2r_1^Tb + t_1b^Tr_2 - r_1^Tj(a)r_2 \\
\tau(X_1)*(M)X_2 &= \begin{pmatrix} t_1 & r_1^T\end{pmatrix}\begin{pmatrix} 0 & b^T \\ b & j(-a)\end{pmatrix}\begin{pmatrix} t_2 \\ r_2\end{pmatrix} = \begin{pmatrix} r_1^Tb & t_1b^T - r_1^Tj(a)\end{pmatrix}\begin{pmatrix} t_2 \\ r_2\end{pmatrix} \\
&= t_2r_1^Tb + t_1b^Tr_2 - r_1^Tj(a)r_2
\end{aligned}$$

(iii) Let us set:

$$M := \begin{pmatrix} 0 & a^T \\ a & j(b)\end{pmatrix} \quad , \quad X := \begin{pmatrix} t \\ r\end{pmatrix} \quad , \quad Y := \begin{pmatrix} t' \\ r'\end{pmatrix}.$$

Since $*(M)X = Y$, we have $\tau(X)Y = 0$ by point (i) of proposition C.2. Thus, we have the four equalities:

$$\begin{aligned}
&(a) \qquad 0 = \tau(X)Y = tt' - r^Tr' \\
&(b) \qquad 1 = \tau(X)X = t^2 - r^Tr \\
&(c) \qquad 0 = MX = \begin{pmatrix} 0 & a^T \\ a & j(b)\end{pmatrix}\begin{pmatrix} t \\ r\end{pmatrix} = \begin{pmatrix} a^Tr \\ ta + b\wedge r\end{pmatrix} \\
&(d) \qquad \begin{pmatrix} t' \\ r'\end{pmatrix} = Y = *(M)X = *\begin{pmatrix} 0 & a^T \\ a & j(b)\end{pmatrix}\begin{pmatrix} t \\ r\end{pmatrix} = \begin{pmatrix} 0 & b^T \\ b & j(-a)\end{pmatrix}\begin{pmatrix} t \\ r\end{pmatrix} = \begin{pmatrix} b^Tr \\ tb - a\wedge r\end{pmatrix}
\end{aligned}$$

We have $t^2 \geq 1$ by point (b) (thus $t \neq 0$), and therefore by points (c) and (d), we have:

$$a = -\frac{1}{t}b\wedge r \quad , \quad b = \frac{1}{t}\left(r' + a\wedge r\right)$$

Since $r^Ta = 0$ (by point (c)), we have with point (b):

$$\begin{aligned}
a &= -\frac{1}{t}b\wedge r = -\frac{1}{t}\left(\frac{1}{t}\left(r' + a\wedge r\right)\right)\wedge r = \frac{1}{t^2}\left(r\wedge r' + r^Tr.a - r^Ta.r\right) \\
&= \frac{1}{t^2}\left(r\wedge r' + \left(t^2-1\right)a\right) = \frac{1}{t^2}\left(r\wedge r' - a\right) + a
\end{aligned}$$

i.e., we have $a = r\wedge r'$. Similarly, we have for b with points (a) and (b):

$$b = \frac{1}{t}\left(r' + \left(r\wedge r'\right)\wedge r\right) = \frac{1}{t}\left(r' + r^Tr.r' - \left(r'\right)^Tr.r\right) = \frac{1}{t}\left(r' + \left(t^2-1\right)r' - tt'r\right) = tr' - t'r$$

Thus, we have:

$$M = \begin{pmatrix} 0 & \left(r\wedge r'\right)^T \\ r\wedge r' & j\left(tr' - t'r\right)\end{pmatrix} = j_4(Y,X).$$

□

We deduce the following corollary on the application j_4.

Corollary C.7

For all $\mathbf{L} \in \mathscr{L}or$ and all $X_1, X_2 \in \mathbb{R}^{1,3}$:

$$j_4(\mathbf{L}X_1, \mathbf{L}X_2) = \det(\mathbf{L})\mathbf{L}j_4(X_1,X_2)\tau(\mathbf{L}).$$

Proof. Let us set $M := X_1\tau(X_2) - X_2\tau(X_1) \in \mathscr{A}(1,3,\mathbb{R})$, and by point (ii) of proposition C.2:

$$\begin{aligned} j_4(\mathbf{L}X_1, \mathbf{L}X_2) &= *(\mathbf{L}X_1\tau(X_2)\tau(\mathbf{L}) - X_2\tau(X_1)\tau(\mathbf{L})) = *(\mathbf{L}M\tau(\mathbf{L})) \\ &= \det(\mathbf{L})\mathbf{L} * (M)\tau(\mathbf{L}) = \det(\mathbf{L})\mathbf{L} * (X_1\tau(X_2) - X_2\tau(X_1))\tau(\mathbf{L}) \\ &= \det(\mathbf{L})\mathbf{L}j_4(X_1, X_2)\tau(\mathbf{L}) \end{aligned}$$

□

C.4 Pfaffian in a Minkowski Space

Definition C.8

Let $M := \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} \in \mathscr{A}(1,3)$. The **pfaffian of** M is the real number:

$$\text{pf}(M) := a^T b.$$

Thus, for $M \in \mathscr{A}(1,3)$, we have:

$$\text{pf}(M) = [M]_{12}[M]_{43} + [M]_{13}[M]_{24} + [M]_{14}[M]_{32}. \tag{108}$$

We have the following simple properties.

Proposition C.9

Let $M \in \mathscr{A}(1,3)$.

(i) We have:
$$\text{pf}(*(M)) = -\text{pf}(M).$$

(ii) We have:
$$\text{pf}(\tau(M)) = -\text{pf}(M).$$

(iii) For all $\alpha \in \mathbb{R}$, we have:
$$\text{pf}(\alpha M) = \alpha^2 \text{pf}(M).$$

(iv) We have:
$$*(M)M = \text{pf}(M) I_4.$$

(v) We have:
$$\det(M) = -\text{pf}(M)^2.$$

Proof. Let $M := \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} \in \mathscr{A}(1,3)$.

(i) We have:
$$\text{pf}(*(M)) = \text{pf}\begin{pmatrix} 0 & b^T \\ b & j(-a) \end{pmatrix} = -a^T b = -\text{pf}(M).$$

(ii) We have:
$$\text{pf}(\tau(M)) = \text{pf}\begin{pmatrix} 0 & -a^T \\ -a & j(b)^T \end{pmatrix} = \text{pf}\begin{pmatrix} 0 & -a^T \\ -a & -j(b) \end{pmatrix} = -\text{pf}(M).$$

(iii) For all $\alpha \in \mathbb{R}$, we have:
$$\text{pf}(\alpha M) = \text{pf}\begin{pmatrix} 0 & \alpha a^T \\ \alpha a & \alpha j(b) \end{pmatrix} = (\alpha a)^T(\alpha b) = \alpha^2 \text{pf}(M).$$

(iv) From point (iii) of lemma A.10:

$$*(M)\,M = \begin{pmatrix} 0 & b^T \\ b & j(-a) \end{pmatrix}\begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} = \begin{pmatrix} b^T a & b^T j(b) \\ j(-a)\,a & ba^T + j(-a)\,j(b) \end{pmatrix} = \begin{pmatrix} b^T a & 0 \\ 0 & b^T a I_3 \end{pmatrix} = b^T a I_4 = \mathrm{pf}(M)\, I_4.$$

(v) Let $a := \sum_{i=1}^{3} a^i e_i$ and $b := \sum_{i=1}^{3} b^i e_i$. We have:

$$\begin{aligned}
&\det(M)\\
&= \det\begin{pmatrix} 0 & a^1 & a^2 & a^3 \\ a^1 & 0 & -b^3 & b^2 \\ a^2 & b^3 & 0 & -b^1 \\ a^3 & -b^2 & b^1 & 0 \end{pmatrix}\\
&= -a^1 \begin{vmatrix} a^1 & -b^3 & b^2 \\ a^2 & 0 & -b^1 \\ a^3 & b^1 & 0 \end{vmatrix} + a^2 \begin{vmatrix} a^1 & 0 & b^2 \\ a^2 & b^3 & -b^1 \\ a^3 & -b^2 & 0 \end{vmatrix} - a^3 \begin{vmatrix} a^1 & 0 & -b^3 \\ a^2 & b^3 & 0 \\ a^3 & -b^2 & b^1 \end{vmatrix}\\
&= -a^1\left(b^1b^3a^3 + b^1b^2a^2 + b^1b^1a^1\right) + a^2\left(-b^2b^2a^2 - b^2b^3a^3 - b^1b^2a^1\right) - a^3\left(b^1b^3a^1 + b^2b^3a^2 + b^3b^3a^3\right)\\
&= -\left(a^1b^1 + a^2b^2 + a^3b^3\right)^2 = -\left(a^Tb\right)^2 = -\mathrm{pf}(M)^2.
\end{aligned}$$

□

Proposition C.10

For all $M \in \mathscr{A}(1,3)$ and any $A \in \mathscr{M}(4,\mathbb{R})$, we have:

$$\mathrm{pf}(AM\tau(A)) = \det(A)\,\mathrm{pf}(M).$$

Proof. Since $\det(\tau(A)) = \det(A)$, we have:

$$-\mathrm{pf}(AM\tau(A))^2 = \det(AM\tau(A)) = \det(A)^2 \det(M) = -\det(A)^2\,\mathrm{pf}(M)^2$$

i.e., we have:

$$(*) \quad : \quad \mathrm{pf}(AM\tau(A)) = \pm\det(A)\,\mathrm{pf}(M).$$

We will prove the result by density and connectedness using matrices with complex coefficients. We extend the application of τ to any matrix $A \in \mathscr{M}(1+k, 1+l, \mathbb{R})$ with $k, l \in \mathbb{N}$ by setting:

$$\tau(A) := I_{1,l} A^T I_{1,k}.$$

We then denote [5]:

$$\mathscr{A}(1,3,\mathbb{C})_{\mathbb{R}} := \{M \in \mathscr{M}(4,\mathbb{C}),\ \tau(M) = -M\}.$$

We show the following lemma.

Lemma C.11

(i) The set $\mathrm{GL}(4,\mathbb{C})$ is dense in $\mathscr{M}(4,\mathbb{C})$.

(ii) The set $\mathscr{A}(1,3,\mathbb{C})_{\mathbb{R}} \cap \mathrm{GL}(4,\mathbb{C})$ is dense in $\mathscr{A}(1,3,\mathbb{C})_{\mathbb{R}}$.

(iii) $\mathrm{GL}(4,\mathbb{C})$ is arc-connected.

[5] We denote this set as such to avoid confusion with the set of anti-Hermitian matrices in a Minkowski space $\mathscr{A}(1,3,\mathbb{C}) := \{M \in \mathscr{M}(4,\mathbb{C}),\ I_{1,3}\overline{A}^T I_{1,3} = -M\}$

Proof. (i) Let $M \in \mathscr{M}(4,\mathbb{C})$. For all $k \in \mathbb{N}_*$, define $B_k := M - \frac{1}{k} I_4$. For all $k \in \mathbb{N}_*$:

$$\det(B_k) = \det\left(M - \frac{1}{k} I_4\right) = \det\left(\frac{1}{k} I_4 - M\right) = \chi_M(1/k)$$

where χ_M is the characteristic polynomial of M of degree 4 (thus admitting at most 4 distinct complex roots). Hence, there exists $k_0 \in \mathbb{N}_*$ such that for all $k \geq k_0$, $\chi_M(1/k) \neq 0$ (otherwise, χ_M would have an infinite number of roots). Therefore, the sequence $(B_{k+k_0})_k$ is a sequence in $\mathrm{GL}(4,\mathbb{C})$ converging to M. Thus, the result.

(ii) Let us define:

$$B := e_1 \ominus e_1 := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

We have $\det(B) = -\mathrm{pf}(B)^2 = -\left(e_1^T e_1\right)^2 = -1$. Therefore, B is invertible, i.e., $B \in \mathscr{A}(1,3,\mathbb{C})_{\mathbb{R}} \cap \mathrm{GL}(4,\mathbb{C})$.

Let $M \in \mathscr{A}(1,3,\mathbb{C})_{\mathbb{R}}$. For all $k \in \mathbb{N}_*$, define $B_k := M - \frac{1}{k} B$. For all $k \in \mathbb{N}_*$:

$$\det(B_k) = \det\left(M - \frac{1}{k} B\right) = \det(-B)\det\left(\frac{1}{k} I_4 - B^{-1} M\right) = \det(B)\,\chi_{B^{-1}M}(1/k)$$

where $\chi_{B^{-1}M}$ is the characteristic polynomial of $B^{-1}M$ of degree 4 (thus admitting at most 4 distinct complex roots). Therefore, there exists $k_0 \in \mathbb{N}_*$ such that for all $k \geq k_0$, $\chi_{B^{-1}M}(1/k) \neq 0$. Since $\det(B) \neq 0$, the sequence $(B_{k+k_0})_k$ is a sequence in $\mathrm{GL}(4,\mathbb{C})$ converging to M. Hence, the result.

(iii) Let $A, B \in \mathrm{GL}(4,\mathbb{C})$. Define $P(X) := \det(XA + (1-X)B)$. Then, P is a polynomial over $\mathbb{C}$ of degree at most 4 (and hence has at most 4 distinct complex roots). Therefore, the open set $\mathscr{U} := \{z \in \mathbb{C},\ P(z) \neq 0\}$ is arc-connected in $\mathbb{C}$. Since $0, 1 \in \mathscr{U}$ (because A, B are invertible), the function

$$g : \begin{array}{ccc} \mathscr{U} & \longrightarrow & \mathrm{GL}(4,\mathbb{C}) \\ z & \longmapsto & zA + (1-z)B \end{array}$$

is continuous, and thus $g(\mathscr{U})$ is arc-connected. Since $A = g(1)$ and $B = g(0) \in g(\mathscr{U})$, we deduce that $\mathrm{GL}(4,\mathbb{C})$ is arc-connected.

□

Let $M \in \mathscr{A}(1,3,\mathbb{C}) \cap \mathrm{GL}(n,\mathbb{C})$. From $(*)$, we have a function:

$$\Phi_M : \begin{array}{ccc} \mathrm{GL}(4,\mathbb{C}) & \longrightarrow & \{\pm 1\} \\ A & \longmapsto & \dfrac{\mathrm{pf}(AM\tau(A))}{\det(A)\,\mathrm{pf}(M)} \end{array}$$

Since $\Phi_M(I_4) = 1$ and $\mathrm{GL}(4,\mathbb{C})$ is connected, the function Φ_M is constant at 1.
Now define:

$$\Psi : \begin{array}{ccc} \mathscr{M}(4,\mathbb{C}) \times \mathscr{A}(1,3,\mathbb{C}) & \longrightarrow & \mathbb{C} \\ (A, M) & \longmapsto & \mathrm{pf}(AM\tau(A)) - \det(A)\,\mathrm{pf}(M) \end{array}$$

Then Ψ is zero on the set $\mathrm{GL}(4,\mathbb{C}) \times (\mathscr{A}(1,3,\mathbb{C}) \cap \mathrm{GL}(4,\mathbb{C}))$, which is dense in $\mathscr{M}(4,\mathbb{C}) \times \mathscr{A}(1,3,\mathbb{C})$. Since Ψ is continuous (because $\Psi(A,M)$ is polynomial in the coefficients of A and M), it is zero on $\mathscr{M}(4,\mathbb{C}) \times \mathscr{A}(1,3,\mathbb{C})$. Hence, the result.

□

Corollary C.12

Let $k \in \mathbb{N}$ and $M \in \mathscr{A}(1,3)$. We have:

$$\mathrm{pf}\left(M^{2k+1}\right) = (-1)^k \,\mathrm{pf}(M)^{2k+1}.$$

Proof. We prove this by induction on $k \in \mathbb{N}$:

$$(\mathscr{P}_k) \ : \ \mathrm{pf}\left(M^{2k+1}\right) = (-1)^k \,\mathrm{pf}(M)^{2k+1}.$$

(i) We have $\mathrm{pf}\left(M^1\right) = \mathrm{pf}(M)^1$. Therefore, $(\mathscr{P}_1)$ is true.

(ii) Let $k \in \mathbb{N}$. Suppose $(\mathscr{P}_k)$ is true. From proposition C.10, we have:

$$\begin{aligned}\mathrm{pf}\left(M^{2k+3}\right) &= \mathrm{pf}\left(MM^{2k+1}M\right) = \det(M)\,\mathrm{pf}\left(M^{2k+1}\right)\\ &= -\mathrm{pf}(M)^2 \times (-1)^k \,\mathrm{pf}(M)^{2k+1} = (-1)^{k+1}\,\mathrm{pf}(M)^{2k+3}.\end{aligned}$$

Therefore, $(\mathscr{P}_{k+1})$ is true.

(iii) By induction, $(\mathscr{P}_k)$ is true for all $k \in \mathbb{N}$.

□

We conclude with the following proposition.

Proposition C.13

Let $X, Y \in \mathbb{R}^{1,3}$ and $M \in \mathscr{A}(1,3)$. Then we have:

$$\mathrm{pf}\left(M + X\tau(Y) - Y\tau(X)\right) = \mathrm{pf}(M) + \tau(Y) * (M)\, X.$$

Proof. Let us define:

$$X := \begin{pmatrix} t \\ r \end{pmatrix} \quad , \quad Y := \begin{pmatrix} t' \\ r' \end{pmatrix} \quad , \quad M := \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix}.$$

We have:

$$*(M)\,X = \begin{pmatrix} 0 & b^T \\ b & j(-a) \end{pmatrix}\begin{pmatrix} t \\ r \end{pmatrix} = \begin{pmatrix} b^T r \\ j(-a)\,r + tb \end{pmatrix} = \begin{pmatrix} b^T r \\ r \wedge a + tb \end{pmatrix}$$

and:

$$\begin{aligned} M + X\tau(Y) - Y\tau(X) &= \begin{pmatrix} 0 & a^T \\ a & j(b) \end{pmatrix} + \begin{pmatrix} t \\ r \end{pmatrix}\begin{pmatrix} t' & -(r')^T \end{pmatrix} - \begin{pmatrix} t' \\ r' \end{pmatrix}\begin{pmatrix} t & -r^T \end{pmatrix}\\ &= \begin{pmatrix} 0 & a^T + t'r^T - t(r')^T \\ a + t'r - tr' & j(b) + r'r^T - r(r')^T \end{pmatrix}\\ &= \begin{pmatrix} 0 & a^T + t'r^T - t(r')^T \\ a + t'r - tr' & j(b + r \wedge r') \end{pmatrix}\end{aligned}$$

Thus we have:

$$\begin{aligned}\mathrm{pf}\left(M + X\tau(Y) - Y\tau(X)\right) &= \left(a^T + t'r^T - t\left(r'\right)^T\right)\left(b + r \wedge r'\right)\\ &= a^T b + t'r^T b - t\left(r'\right)^T b + a^T\left(r \wedge r'\right) + \left(t'r^T - t\left(r'\right)^T\right)\left(r \wedge r'\right)\\ &= a^T b + t'b^T r - t\left(r'\right)^T b - \left(r'\right)^T (r \wedge a)\\ &= \mathrm{pf}(M) + \begin{pmatrix} t' & -(r')^T \end{pmatrix}\begin{pmatrix} b^T r \\ r \wedge a + tb \end{pmatrix}\\ &= \mathrm{pf}(M) + \tau(Y) * (M)\, X\end{aligned}$$

□

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