A new proof of Birkhoff's theorem*

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Abstract

Assuming $SO(3)$-spherical symmetry, the 4–dimensional Einstein equation reduces to an equation conformally related to the field equation for 2–dimensional gravity following from the Lagrangian $L = |R|^{1/3}$.

Solutions for 2–dimensional gravity always possess a local isometry because the traceless part of its Ricci tensor identically vanishes.

Combining both facts, we get a new proof of Birkhoff's theorem; contrary to other proofs, no coordinates must be introduced.

The $SO(m)$-spherically symmetric solutions of the $(m+1)$–dimensional Einstein equation can be found by considering $L = |R|^{1/m}$ in two dimensions. This yields several generalizations of Birkhoff's theorem in an arbitrary number of dimensions, and to an arbitrary signature of the metric.

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1 Introduction

The Birkhoff theorem states that every spherically symmetric vacuum solution of Einstein's general relativity theory is part of the Schwarzschild solution; therefore, the solution possesses a four-dimensional[1] isometry group, at least locally. This theorem has been generalized into several directions, e.g. to Einstein–Maxwell fields and to higher dimensions by Bronnikov, Kovalchuk and Melnikov in [1].

In the meanwhile it is generally known that the popular formulation "every spherically symmetric vacuum solution is static" is misleading, because inside the horizon, the Schwarzschild black hole is not static. Now, Bondi and Rindler [2] argued, that even in the absence of a horizon, the use of the word "static" might be misleading, too.

Both Schmutzer and Goenner [3] cite refs. [4] for the original Birkhoff theorem, the historic development can be summarized as follows: "Jebsen (1921) was the first to formulate it and Birkhoff (1923) was the first to prove it". Hawking, Ellis [5, page 369] gave a proof of it based on B. Schmidt's method [6]. A further proof is given in sct. 32.2 of Misner, Thorne and Wheeler [7]. They relate this theorem to the fact that no monopole gravitational waves exist in Einstein's theory. The Birkhoff theorem in 2–dimensional pure metric gravity has been deduced in [8] and generalized in [9]. It possesses analogous formulations for 2-dimensional dilaton–gravity theories, cf. refs. [10 - 14] for its recent development.

In ref. [15], Ashtekar, Bicak and B. Schmidt considered those solutions of the 4–dimensional Einstein equation which possess one translational Killing field. The non-trivial solutions fail to be asymptotically flat; to get nevertheless a sensible notion of energy it turned out to be advantageous to reformulate the problem as 3–dimensional gravity with additional matter.

[1]This is the remarkable part of that theorem, because the rotation group $SO(3)$ is only 3–dimensional.

At the same formal level, but with another scope as in [15], Romero, Tavakol and Zalaletdinov [16] reduced the dimension from 5 to 4. One of their results read: 4–dimensional gravity with matter can be isometrically embedded into a 5–dimensional Ricci–flat space–time, i.e., into a vacuum solution of the 5–dimensional Einstein equation.

In the present paper, we follow a similar line as in [15] and [16] and deduce Birkhoff–type theorems for warped products of manifolds where one of them is two–dimensional. (Here we use the notion "Birkhoff–type theorem" for any theorem stating that under certain circumstances the gravitational vacuum solution has more symmetries than the inserted metric ansatz.) The proof is done in two steps: First, the D–dimensional Einstein equation will be reduced to two–dimensional gravity. Second: By applying the fact that the traceless part of the Ricci tensor identically vanishes in two dimensions, we always get an additional Killing vector.

As a byproduct, we present a new and coordinate–free proof of the classical Birkhoff theorem.

The paper is organized as follows: In sct. 2 the warped products are introduced, in sct. 3 the necessary conformal transformation is explained, and in sct. 4 the new direct proof of the Birkhoff theorem is given by an explicit definition of the additional Killing vector. Sct. 5 shows the detailed relation to several 2-dimensional theories, and sct. 6 gives the summary.

2 Warped products

Let us consider a D-dimensional Riemannian manifold of arbitrary signature and metric ds^2 of the form

$$ds^2 \;=\; g_{\alpha\beta}dx^\alpha dx^\beta \;=\; d\sigma^2 \;+\; e^{2U}d\hat{\Omega}^2 \qquad (2.1)$$

where

$$d\sigma^2 \;=\; g_{ij}dx^i dx^j \qquad (2.2)$$

is a two-dimensional manifold and

$$d\hat{\Omega}^2 \;=\; \hat{g}_{AB}dx^A dx^B \qquad (2.3)$$

is n–dimensional with $n \geq 1$. Hence, $D = n + 2 \geq 3$. The indices i, j take the values 0, 1; the indices A, B take the values 2, ..., $n+1$; and the indices α, β cover both of them, i.e., values 0, ..., $n+1$.

We assume that $\hat{g}_{AB}$ depends on the x^A only, and both U and g_{ij} depend on the x^i only. So we have defined ds^2 to be the warped product between $d\sigma^2$ and $d\hat{\Omega}^2$ with warping function e^{2U}. **The purpose of the whole consideration is to show the following: If ds^2 is a D-dimensional Einstein space and the dimension of the isometry group of $d\hat{\Omega}^2$ equals k, then ds^2 possesses a $k+1$–dimensional isometry group, at least locally.** Taking $d\hat{\Omega}^2$ as standard two–sphere, we recover the original Birkhoff theorem.

The method we want to apply for proving this is first to reformulate the D-dimensional Einstein equation for ds^2 as a field equation in the 2-dimensional space $d\sigma^2$, and second to apply known results [8, 9] from scale–invariant gravity in 2 dimensions.

3 A conformal transformation

To simplify the calculation of the Ricci tensor of ds^2, we perform a conformal transformation in D dimensions as follows:

$$d\hat{s}^2 \;=\; \hat{g}_{\alpha\beta} dx^\alpha dx^\beta \;=\; e^{-2U}\; ds^2 \tag{3.1}$$

Using eq. (2.1) this implies

$$d\hat{s}^2 \;=\; d\hat{\sigma}^2 \;+\; d\hat{\Omega}^2 \tag{3.2}$$

where the analogous conformal transformation in 2 dimensions reads

$$d\hat{\sigma}^2 \;=\; e^{-2U}\; d\sigma^2 \tag{3.3}$$

By construction,

$$g_{\alpha\beta} \;=\; e^{2U}\; \hat{g}_{\alpha\beta} \tag{3.4}$$

and

$$\hat{g}_{iA} = 0 \qquad \hat{g}_{ij,A} = 0 \qquad \hat{g}_{AB,i} = 0 \tag{3.5}$$

We denote the D–dimensional Ricci tensor for ds^2 by ${}^{(D)}R_{\alpha\beta}$, and the 2–dimensional Ricci tensor for $d\sigma^2$ by R_{ij}. For the hatted quantities, we need not the distinguishing presuperscript because by eqs. (3.2), (3.5), the Ricci tensor $\hat{R}_{ij}$ for $d\hat{\sigma}^2$ and the Ricci tensor $\hat{R}_{AB}$ for $d\hat{\Omega}^2$ together form the Ricci tensor $\hat{R}_{\alpha\beta}$ for $d\hat{s}^2$. The latter fulfils $\hat{R}_{Ai} = 0$ because $d\hat{s}^2$ represents a direct product.

The Ricci scalars are denoted as follows: ${}^{(D)}R$ for ds^2; R for $d\sigma^2$; ${}^{(D)}\hat{R}$ for $d\hat{s}^2$; $\hat{R}$ for $d\hat{\sigma}^2$; and ${}^{(n)}\hat{R}$ for $d\hat{\Omega}^2$. Applying that $d\hat{s}^2$ is a direct product we get

$${}^{(D)}\hat{R} \;=\; \hat{R} \;+\; {}^{(n)}\hat{R} \tag{3.6}$$

The assumption that ds^2 is an Einstein space reads

$${}^{(D)}R_{\alpha\beta} \;=\; \Lambda\; g_{\alpha\beta} \tag{3.7}$$

where $\Lambda = \frac{1}{D}\ {}^{(D)}R$ and has a constant value because of $D \geq 3$.

$\Box$ denotes the 2-dimensional D'Alembertian in $d\sigma^2$, and $\hat{\ }\Box$ the analogous operator in $d\hat{\sigma}^2$. Eqs. (2.2) and (3.3) imply that

$$\hat{g}_{ij} \ = \ e^{-2U}\ g_{ij} \tag{3.8}$$

is the metric for $d\hat{\sigma}^2$. Then for the Ricci tensors the following relation holds:

$$\hat{R}_{ij} \ = \ R_{ij} \ + \ g_{ij}\,\Box U \tag{3.9}$$

i.e.

$$\hat{R} \ = \ (R \ + \ 2\Box U)\, e^{2U} \tag{3.10}$$

The conformal invariance of the 2-dimensional D'Alembertian implies

$$\hat{\ }\Box U \ = \ e^{2U}\ \Box U \tag{3.11}$$

therefore, eq. (3.10) can be rewritten as

$$R \ = \ (\hat{R} \ - \ 2\,\hat{\ }\Box U)\, e^{-2U} \tag{3.12}$$

Now, we turn to the D-dimensional conformal transformation eq. (3.4), we get

$${}^{(D)}R_{\alpha\beta} = \hat{R}_{\alpha\beta} - n[U_{\hat{;}\alpha\beta} - U_{;\alpha}U_{;\beta}] - \hat{g}_{\alpha\beta}[\hat{\ }\Box U + nU_{;\gamma}U^{\hat{;}\gamma}] \tag{3.13}$$

where the "$\hat{;}$" denotes the covariant derivative made with $\hat{g}_{\alpha\beta}$ and

$$U^{\,\hat{;}\,\gamma} \ = \ \hat{g}^{\,\gamma\,\mu}\ U_{;\,\mu}$$

First we insert eq. (3.7) into eq. (3.13) and consider the AB-part of it only. We get after some algebra

$$\hat{R}_{AB} \ = \ \hat{\Lambda}\ \hat{g}_{AB} \tag{3.14}$$

where we have applied the abbreviation

$$\hat{\Lambda} \;=\; \Lambda e^{2U} \;+\; \hat{}\Box U \;+\; n\, U_{;\gamma}\, U^{\hat{;}\gamma} \tag{3.15}$$

By construction, $\hat{R}_{AB}$ and $\hat{g}_{AB}$ depend on the x^A only, whereas $\hat{\Lambda}$ depends on the x^i only. Consequently, $\hat{\Lambda}$ must be a constant. Remark: For $n \geq 3$, this statement follows already from the structure of eq. (3.14). For $n = 2$, however, it is not obvious from the beginning. Eq. (3.14) expresses the fact that $d\hat{\Omega}^2$ must be a n–dimensional Einstein space.

Transvecting eq. (3.14) with $\hat{g}^{AB}$ we get

$$^{(n)}\, \hat{R} \;=\; n\, \hat{\Lambda} \tag{3.16}$$

In the next step, we consider the i, j–part of eq. (3.13). We get

$$\Lambda g_{ij} = \hat{R}_{ij} - n[U_{\hat{;}ij} - U_{;i}U_{;j}] - \hat{g}_{ij}[\,\hat{}\Box U + nU_{;m}U^{\hat{;}m}] \tag{3.17}$$

Transvecting eq. (3.17) with $\hat{g}^{ij}$ we get

$$2\,\Lambda\, e^{2U} \;=\; \hat{R} \;-\; D\,\hat{}\Box\, U \;-\; n\, U_{;i}\, U^{\hat{;}i} \tag{3.18}$$

We apply eqs. (3.10), (3.11) and use the notation

$$U^{\,,\,i} \;=\; g^{\,i\,j}\; U_{,\,j}$$

Then we get

$$R \;=\; 2\,\Lambda \;+\; n\,(\,\Box\, U \;+\; U_{,i}\, U^{,i}\,) \tag{3.19}$$

The remaining of eq. (3.17) consists in the requirement that the traceless part of the tensor

$$\hat{H}_{ij} \;=\; U_{\hat{;}ij} - U_{;i}U_{;j} \tag{3.20}$$

has to vanish.

4 The 2-dimensional picture (direct way)

In this section, we consider gravity in two dimensions in the two conformally related frames $d\sigma^2$ and $d\hat{\sigma}^2$, and the additional scalar field U. The essential equations are (3.15), (3.18) and (3.20): They represent the field equations in the hatted frame. To get the equations in the unhatted frame we use eqs. (3.3) and (3.8). As a result, we have to replace (3.18) by (3.19), (3.15) by the equation

$$\Lambda \;=\; \hat{\Lambda} e^{-2U} \;-\; \Box U \;-\; n\, U_{,i}\, U^{,i} \tag{4.1}$$

The tracelessness of the term (3.20) will be transformed to the condition that

$$H_{ij} \;=\; U_{;ij} + U_{;i} U_{;j} \tag{4.2}$$

is traceless. One can rewrite eq. (4.2) as

$$H_{ij} \;=\; e^{-U} \; (\, e^{U} \,)_{;ij}$$

so that the condition is equivalent to say that the traceless part of $(\, e^{U} \,)_{;ij}$ must vanish.

At this point we are able to apply appendix B of ref. [8], where e^U now plays the role of G eq. (B2) of [8].

First case[2] : Let U be constant in any region of space. By eq. (4.1) this case appears only if Λ and $\hat{\Lambda}$ have the same sign. Eq. (3.19) then implies $R = 2\Lambda$. Consequently, $d\sigma^2$ represents a 2–surface of constant curvature, and the warped product eq. (2.1) specializes to a direct product. So we get for this first case: If the dimension of the isometry group of $d\hat{\Omega}^2$ equals k, then ds^2 possesses a $k+3$–dimensional isometry group, at least locally.

Second case: Let $U_{;\,i} \neq 0$ in any region of space. We choose an orientable subregion W of it and fix an orientation there. Then we are allowed to

[2] i.e., that case where Schwarzschild coordinates cannot be introduced for a spherically symmetric metric

use the covariantly constant antisymmetric pseudotensor ϵ_{ij} within W. It is completely defined by fixing the component

$$\epsilon_{01} = \sqrt{|\det g_{ij}|}$$

Indices will be shifted with the metric g_{ij}, and then we define

$$\xi_i \;=\; \epsilon_{ij}\; (\, e^U \,)^{;j} \tag{4.3}$$

which is a nowhere vanishing vector field in the region W. [In index–free notation this can be written as curl(e^U) which is analogous to the rot-operator in three dimensions.] Let us calculate its covariant derivative.

$$\xi_{i;k} \;=\; \epsilon_{ij}\; (\, e^U \,)^{;j}_{;k}$$

Because of eq. (4.2) $(\, e^U \,)^{;j}_{;k}$ is c times the Kronecker tensor. Therefore, we get

$$\xi_{i;k} \;=\; \epsilon_{ij}\; c\; \delta^j_k \;=\; c\; \epsilon_{ik} \tag{4.4}$$

The antisymmetry of the ϵ–pseudotensor proves ξ_i to be a Killing vector in $d\sigma^2$.

But, in contrast to the first case, here it is not immediately clear that an isometry of $d\sigma^2$ represents an isometry of ds^2.

Subcase 2.1.: Let ξ_i be a non–vanishing light-like vector in any region of space. Then by sct. V A of ref. [8] $d\sigma^2$ is flat and of signature $(+-)$. It turns out that one of its Killing vectors (not necessarily ξ_i itself) represents also a Killing vector of ds^2.

Subcase 2.2.: Let ξ_i be a non–light–like vector in any region of space. Then, by construction, it is tangential to the lines of constant values U. Therefore ξ_i represents also a Killing vector of ds^2.

In both subcases, we get: If the dimension of the isometry group of $d\hat{\Omega}^2$ equals k, then ds^2 possesses a $k+1$–dimensional isometry group, at least locally.

The boundaries of the cases 1 and 2.1., 2.2., can be covered by a continuity argument.

5 The 2-dimensional picture (way via fourth-order gravity)

In this section, we consider gravity in two dimensions in the two conformally related frames $d\sigma^2$ and $d\tilde{\sigma}^2$, and the additional scalar field U. The three essential equations are (3.19), (4.1) and (4.2). We repeat them here for convenience.

$$R \;=\; 2\,\Lambda \;+\; n\,(\,\Box\,U \;+\; U_{,i}\,U^{,i}\,) \tag{5.1}$$

$$\Lambda \;=\; \hat{\Lambda} e^{-2U} \;-\; \Box U \;-\; n\,U_{,i}\,U^{,i} \tag{5.2}$$

and the traceless part of

$$(e^U)_{;ij} \tag{5.3}$$

has to vanish. The idea is to write the D-dimensional Einstein–Hilbert action for the metric ansatz eq. (2.1). Eqs. (3.6, 3.12, 3.16) can be subsumed to

$$^{(D)}\,R = R + n\hat{\Lambda}e^{-2U} - n(n+1)U_{,i}U^{,i} - 2n\Box U \tag{5.4}$$

The 2–dimensional Lagrangian is obtained after integrating the n-dimensional part; the gradient of U can be eliminated from (5.4) by applying the fact that

$$(e^{nU}\,U^{,i})_{;i} = e^{nU}(\Box U + nU^{,i}U_{,i})$$

represents a divergence. Then we get the density

$$\mathcal{L} = [R - (n-1)\Box U - n\Lambda + n\hat{\Lambda}e^{-2U}]e^{nU}\sqrt{|g|} \tag{5.5}$$

where $g = \det g_{ij}$ is the two-dimensional determinant, and e^{nU} is left over from the n-dimensional integration. For $n = 1$ we can take this form as it stands, but for $n > 1$ we apply a conformal transformation

$$\tilde{g}_{ij} = e^{(n-1)U} g_{ij} \tag{5.6}$$

to remove the term with $\Box U$. Analogously to eqs. (3.8), (3.12) we get

$$\tilde{R} = [R - (n-1)\Box U] e^{-(n-1)U} \tag{5.7}$$

and the corresponding Lagrangian scalar

$$\tilde{L} = e^{nU} \tilde{R} - n\Lambda e^{U} + n\hat{\Lambda} e^{-U} \tag{5.8}$$

fulfils $\mathcal{L} = \tilde{L}\sqrt{|\tilde{g}|}$, and it holds: Variation of this $\mathcal{L}$ with respect to U and $\tilde{g}_{ij}$ gives a system of equations equivalent to eqs. (5.1, 5.2, 5.3). In the next step (subsection 5.2.), we try to write this as one single Lagrangian depending on the two–dimensional metric only, i.e., we want to eliminate the scalar U, and this is possible if one takes a Lagrangian non-linear in $\tilde{R}$ as will be shown in subsection 5.1.

5.1 Transformation between fourth and second order

Here we give a more complete description of that transformation which was sketched in (2.4), (2.5) of ref. [9] already. Let $L = L(R)$ be a non–linear Lagrangian in two dimensions, i.e., $G = \frac{dL}{dR} \neq 0$ (without loss of generality we assume $G > 0$) and $\frac{dG}{dR} \neq 0$. The fourth–order field equation following from the variation of $L\sqrt{|g|}$ with respect to g_{ij} has the trace

$$0 = GR - L + \Box G \tag{5.9}$$

and the trace–free part of $G_{;ij}$ must vanish. (For dimension 3 or higher, one must instead ensure the vanishing of the traceless part of $G_{;ij} - GR_{ij}$ which is not an equivalent requirement, see e.g. ref. [17]. Therefore, the key of the proof that eq. (4.3) represents a Killing vector is seen to rest on the dimension 2 where the traceless part of R_{ij} automatically vanishes.)

Now, we introduce a scalar field φ by $G = e^{-2\varphi}$. We invert this relation to $R = R(\varphi)$ which is (at least locally) possible because of our assumptions. Then we define

$$V(\varphi) \; = \; e^{-2\varphi} R(\varphi) \; - \; L(R(\varphi)) \tag{5.10}$$

and the Lagrangian can now be written as

$$L(\varphi, \, R) \; = \; e^{-2\varphi} \, R \; - \; V(\varphi) \tag{5.11}$$

If we take this $L(\varphi, \, R)$ as new starting point, then the field equations become equivalent: variation with respect to φ simply reads

$$0 \; = \; 2 \, e^{-2\varphi} R \; + \; \frac{dV}{d\varphi} \tag{5.12}$$

The variation of $L\sqrt{|g|}$ with respect to the metric gives the trace

$$0 \; = \; V(\varphi) \; + \; \Box(e^{-2\varphi}) \tag{5.13}$$

and the traceless part of $(e^{-2\varphi})_{;jk}$ vanishes. The two second–order eqs. (5.12, 5.13) on the one hand and the single fourth–order eq. (5.9) on the other hand are equivalent. [Let us add the following: We may overcome the singular point $G = 0$ by observing that L and $L + \alpha R$ for a constant α give rise to the same fourth–order field equation; the related G will be increased by α, and the field φ changes in a non–linear manner.]

Now we go the other direction: Let $V(\varphi)$ be given and we take the Lagrangian (5.11). By eq. (5.12) we calculate $R = R(\varphi)$. First case: This R is a constant. This takes place if $L = e^{-2\varphi}(R - \Lambda)$ with any constant Λ. This is the case where the transformation to fourth–order gravity becomes impossible.

Second case: This R is not a constant function. This takes place if the potential V fulfils the inequality

$$\frac{d^2V}{d\varphi^2} \; + \; 2\frac{dV}{d\varphi} \; \neq \; 0 \tag{5.14}$$

Then we can invert to $\varphi = \varphi(R)$ and insert this into $L(R) = L(\varphi(R), R)$. This is the desired non–linear Lagrangian leading to a fourth-order field equation equivalent that one following from eq. (5.11).

5.2 Application of this transformation

The transformation given in subsection 5.1. shall now be applied to the system eq. (5.8) in the tilted version. Eqs. (5.8) and (5.11) get the same structure if we put $U = -2\varphi/n$ and

$$V(\varphi) \; = \; n\Lambda e^{-2\varphi/n} \; - \; n\hat{\Lambda} e^{2\varphi/n} \tag{5.15}$$

Then the tilted version of eq. (5.12) leads to

$$\tilde{R}(\varphi) \; = \; e^{2\varphi}[\Lambda e^{-2\varphi/n} + \hat{\Lambda} e^{2\varphi/n}] \tag{5.16}$$

First case: This value $\tilde{R}$ is constant. (For $n = 1$, we have always $\hat{\Lambda} = 0$, so that $n = 1$ is always subsumed under this first case.) Then the transformation to fourth-order gravity is impossible.

Second case: $n \geq 2$ and either Λ or $\hat{\Lambda}$ is non–vanishing. Then inequality (5.14) is fulfilled and the fourth–order Lagrangian can be given, but not always in closed form. Therefore we restrict to two typical examples:

First example: $\Lambda = 0$ and $\hat{\Lambda} = 1$. We get the Lagrangian

$$\tilde{L} \; = \; (n+1)\, \tilde{R}^{1/(n+1)} \tag{5.17}$$

In such a context, this Lagrangian was first deduced by Rainer and Zhuk, ref. [18, eq. (3.9)].

Second example: $\Lambda = 1$ and $\hat{\Lambda} = 0$. We get

$$\tilde{L} \;=\; (1-n)\,\tilde{R}^{1/(1-n)} \tag{5.18}$$

i.e., in both cases we get scale-invariant fourth-order gravity where the Birkhoff theorem and the corresponding solutions are known.

6 Summary

We discussed the warped product metric ansatz (cf. eqs. (2.1), (5.6))

$$ds^2 \;=\; e^{(1-n)U(x^i)} d\tilde{\sigma}^2 \;+\; e^{2U(x^i)} d\hat{\Omega}^2 \tag{6.1}$$

where

$$d\tilde{\sigma}^2 = \tilde{g}_{ij}(x^k)dx^i dx^j \qquad i,j,k = 0,1$$

is a two–dimensional metric and

$$d\hat{\Omega}^2 = \hat{g}_{AB}(x^C)dx^A dx^B \qquad A,B,C = 2,\dots n+1$$

is n-dimensional, $n \geq 1$. The question was: Under which circumstances eq. (6.1) represents an Einstein space in $D = n+2$ dimensions ? First, $d\hat{\Omega}^2$ has to be an Einstein space in n dimensions, i.e., eq. (3.14) has to be fulfilled with $\hat{\Lambda} = const.$ (it must vanish for $n = 1$). Second, we introduced the scalar field φ via eq. (5.15)

$$U = -2\varphi/n \tag{6.2}$$

and got an equation of 2-dimensional dilaton gravity. The latter can be rewritten as fourth–order gravity in two dimensions.

To ease reading, we restrict now to the case $\Lambda = 0$ and $\hat{\Lambda} = 1$, i.e., ds^2 is Ricci-flat, and for $d\hat{\Omega}^2$, metric and Ricci tensor coincide (i.e. $n \geq 2$). Then eq. (5.16) implies

$$\tilde{R} = e^{2\varphi(n+1)/n} \tag{6.3}$$

Inserting eqs. (6.2), (6.3) into eq. (6.1) we get

$$ds^2 \; = \; \tilde{R}^{(n-1)/(n+1)} d\tilde{\sigma}^2 \; + \; \tilde{R}^{-2/(n+1)} d\hat{\Omega}^2 \tag{6.4}$$

Under these circumstances it holds: Metric (6.4) is D-dimensional Ricci flat if $d\tilde{\sigma}^2$ is a solution of the 2-dimensional fourth–order scale–invariant gravity following from the Lagrangian (5.17) $\tilde{L} = (n+1)\tilde{R}^{1/(n+1)}$ with $\tilde{R} \neq 0$. (If $\tilde{R} < 0$, we have to write $|\tilde{R}|$ instead of $\tilde{R}$.) And the solutions of 2-dimensional fourth–order scale–invariant gravity are known in closed form. So, we have not only proven a generalized Birkhoff theorem for such a warped product of manifolds, but we have also given a procedure how the solutions can be found in closed form.

To show how this procedure works, let us deduce the spherically symmetric vacuum solution of Einstein's theory in 4 dimensions. To this end we put $n = 2$ into eq. (6.4) and take $d\hat{\Omega}^2 = d\psi^2 + \sin^2 \psi d\Phi^2$ as standard two–sphere. (That every spherically symmetric 4-metric can be written this way with metric (2.1) is not proven here, but cf. e.g. ref. [6] for this point.) Eq. (5.17) reads $\tilde{L} = 3\tilde{R}^{1/3}$, this is the case $k = -2/3$ in the notation of ref. [8], hence, by eqs. (19, 22) of [8] we get

$$d\tilde{\sigma}^2 = \frac{dw^2}{A(w)} - A(w) dy^2 \tag{6.5}$$

with $A(w) = C + E\sqrt{w}$ and $\tilde{R} = \frac{E}{4} w^{-3/2}$. We put $E = 4$, and with eqs. (6.4), (6.5) we get

$$ds^2 \; = \; \frac{1}{\sqrt{w}} \left[\frac{dw^2}{C + 4\sqrt{w}} - (C + 4\sqrt{w}) dy^2 \right] \; + \; w d\hat{\Omega}^2 \tag{6.6}$$

With $w = r^2$, $y = t/2$, $C = -8m$ this leads to

$$ds^2 \; = \; \frac{dr^2}{1 - \frac{2m}{r}} \; - \; (1 - \frac{2m}{r}) dt^2 \; + \; r^2 [d\psi^2 + \sin^2 \psi d\Phi^2] \tag{6.7}$$

i.e., the correct Schwarzschild solution. (The generalization to higher dimension n is straightforwardly done.)

The transformation presented here is of a similar structure to that one presented in [19]; however, in [19] the two–dimensional case for $d\sigma^2$ was excluded, and here we solely restricted to that two–dimensional case. In this sense, both approaches are disjoint. The ansatz eq. (4.3) as Killing vector for 2–dimensional models has already be applied several times, e.g. in refs. [8] and [20].

It should be mentioned that at no place the signature of space-time was used, so the Birkhoff theorem is shown to be valid in any signature. The direct way in sct. 4 is equivalent to the way via fourth-order gravity in sct. 5. For both ways the whole proof was done in a fully covariant manner, i.e., no special coordinates had to be introduced. So we also circumvented the discussion whether the introduction of Schwarzschild coordinates represents a loss of generality or not.[3] The key element of the proof follows from the fact that the traceless part of the Ricci tensor in two dimensions identically vanishes, i.e., for the scalar U, the traceless part of $(e^U)_{;ij}$ must vanish. As a consequence, $\xi_i = \epsilon_{ij} (e^U)^{;j}$ could be proven to be a Killing vector.

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References

[1] V. Ruban, Abstracts Conf. General Relativity GR 8 Waterloo, 303 (1977); K. Bronnikov and M. Kovalchuk, *J. Phys. A: Math. Gen.* **13**, 187 (1980); K. Bronnikov and V. Melnikov, *Gen. Relat. Grav.* **27**, 465 (1995).

[2] H. Bondi and W. Rindler, *Gen. Relat. Grav.* **29**, 515 (1997).

[3] By the way, a direct product between two 2-spaces of constant curvature represents an example of a spherically symmetric solution of Einstein's equation with $\Lambda > 0$ which cannot be written in Schwarzschild coordinates.

[3] E. Schmutzer, “Relativistische Physik”, Teubner-Verlag, Leipzig 1968; H. Goenner, *Commun. Math. Phys.* **16**, 34 (1970).

[4] G. Birkhoff and R. Langer, “Relativity and modern physics”, Harvard University Press, Boston 1923; J. Jebsen, *Ark. Mat. Astron. Fys.* **15**, 18 (1921); W. Alexandrov, *Ann. d. Phys. (Leipz.)* **72**, 141 (1923).

[5] S. Hawking and G. Ellis, “The large scale structure of space–time”, Cambridge University Press 1973.

[6] B. Schmidt, *Zeitschr. f. Naturforsch.* **22a**, 1351 (1967).

[7] C. Misner, K. Thorne and J. Wheeler, “Gravitation”, Freeman, San Francisco 1973.

[8] H.-J. Schmidt, *J. Math. Phys.* **32**, 1562 (1991).

[9] S. Mignemi and H.-J. Schmidt, *Class. Quant. Grav.* **12**, 849 (1995).

[10] S. Mignemi, *Ann. Phys. (NY)* **245**, 23 (1996).

[11] J. Creighton and R. Mann, *Phys. Rev.* **D 54**, 7476 (1996).

[12] R. Balbinot and A. Fabbri, *Class. Quant. Grav.* **13**, 2457 (1996).

[13] M. Katanaev, W. Kummer and H. Liebl, *Phys. Rev.* **D 53**, 5609 (1996).

[14] T. Klösch and T. Strobl, *Class. Quant. Grav.* **13**, 965 (1996).

[15] A. Ashtekar, J. Bicak and B. Schmidt, *Phys. Rev.* **D 55**, 669 (1997).

[16] C. Romero, R. Tavakol and R. Zalaletdinov, *Gen. Relat. Grav.* **28**, 365 (1996).

[17] H.-J. Schmidt, gr-qc/9703002; *Gen. Relat. Grav.* **29**, 859 (1997).

[18] M. Rainer and A. Zhuk, *Phys. Rev.* **D 54**, 6186 (1996).

[19] H.-J. Schmidt, *Int. J. Mod. Phys.* **A 5**, 4661 (1990).

[20] T. Banks and M. Loughlin, *Nucl. Phys.* **B 362**, 649 (1991); S. Solodukhin, *Phys. Rev.* **D 51**, 591 (1995).